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On the *p*-Median Polytope and the Directed Odd Cycle Inequalities II: Oriented Graphs

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ON THE p-MEDIAN POLYTOPE AND THE DIRECTED ODD CYCLE INEQUALITIES II: ORIENTED GRAPHS

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Abstract. This is the second part of a study of the odd directed cycle inequalities in the description of the polytope associated with the p-median problem. We treat oriented graphs, i.e., if (u,v) is in the arc-set, then (v,u) is not in the arc-set. We characterize the oriented graphs for which the obvious linear relaxation together with the directed odd cycle inequalities describe the p-median polytope. In the first part [2], we treated triangle-free graphs, this is the first step for an induction on the number of triangles used in the present paper to treat general oriented graphs.

1. Introduction

Let $G = (V, A)$ be a directed graph, not necessarily connected, where each arc $(u, v) \in A$ has an associated cost $c(u, v)$. The *p-median problem* (pMP) consists of selecting p nodes, usually called *centers*, and then assign each nonselected node to a selected node. The goal is to select p nodes that minimize the sum of the costs yielded by the assignment of the nonselected nodes.

If we associate the variables y to the nodes, and the variables x to the arcs, the following is a linear relaxation of the pMP.

$$\sum_{v \in V} y(v) = p, \tag{1}$$

$$y(u) + \sum_{v:(u,v) \in A} x(u,v) = 1 \quad \forall u \in V, \tag{2}$$

$$x(u,v) \le y(v) \quad \forall (u,v) \in A, \tag{3}$$

$$0 \le y(v) \quad \forall v \in V, \tag{4}$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in A. \tag{5}$$

$$\sum_{a \in A(C)} x(a) \le \frac{|A(C)| - 1}{2} \quad \text{for each odd directed cycle } C. \tag{6}$$

We denote by $P_p(G)$ the polytope defined (1)-(5), and by $PC_p(G)$ the polytope defined by (1)-(6). Also let $pMP(G)$ be the convex hull of $P_p(G) \cap \{0,1\}^{|V|+|A|}$. The *p-median polytope* of a graph G is $pMP(G)$. In general we have

$$pMP(G) \subseteq PC_p(G) \subseteq P_p(G).$$

The purpose of this study is to characterize the oriented graphs G for which

$$pMP(G) = PC_p(G).$$

A simple *cycle* C is an ordered sequence $v_0, a_0, v_1, a_1, \ldots, a_{t-1}, v_t$, where

- v_i, $0 \le i \le t-1$, are distinct nodes,
- either v_i is the tail of a_i and v_{i+1} is the head of a_i, or v_i is the head of a_i and v_{i+1} is the tail of a_i, for $0 \le i \le t-1$, and
- $v_0 = v_t$.

Let $V(C)$ and $A(C)$ denote the nodes and the arcs of C, respectively. By setting $a_t = a_0$, we associate with C three more sets : $\hat{C}$, $\dot{C}$ and $\tilde{C}$. Each node v is incident to two arcs a' and a'' of C. If v is the head (resp. tail) of both arcs a' and a'' then v is in $\hat{C}$ (resp. $\dot{C}$) and if v is the head of one of them and a tail of the other, then v is in $\tilde{C}$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle will be called *g-odd* if $|\tilde{C}| + |\hat{C}|$ is odd, otherwise it will be called *g-even*. A cycle C with $V(C) = \tilde{C}$ is a *directed* cycle, otherwise it is called a *non-directed* cycle. The notion of g-odd (g-even) cycles generalizes the notion of odd (even) directed cycles. The *size* of C is $|V(C)|$. A directed cycle of size three will be called a *triangle*.

Definition 1. *A simple cycle is called a Y-cycle if for every $v \in \hat{C}$ there is an arc $(v, \bar{v})$ in A, where $\bar{v}$ is in $V \setminus \dot{C}$.*

The main theorem of this paper is

Theorem 2. *Let $G = (V, A)$ be an oriented graph, then $PC_p(G)$ is integral for any integer p if and only if*

(C1) *it does not contain as a subgraph neither of the graphs H_1, H_2 nor H_3 of Figure 1, and*
(C2) *it does not contain a non-directed g-odd Y-cycle C and an arc (u, v) with neither u nor v in $V(C)$.*

The graphs H_1, H_2 and H_3 are defined in Figure 1 below.

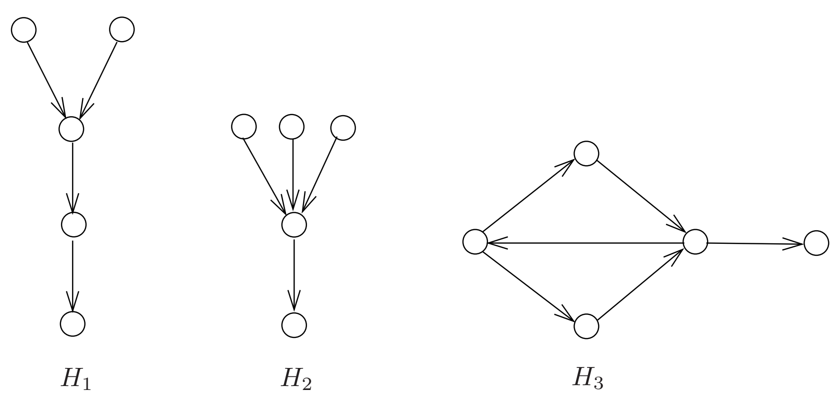

FIGURE 1

In [2] we proved the following two results.

Theorem 3. *Let $G = (V, A)$ be an oriented graph. If G satisfies condition (C1) of Theorem 2 and does not contain a non-directed g-odd Y-cycle, then $PC_p(G)$ is integral, for any integer p.*

Theorem 4. *Let $G = (V, A)$ be an oriented graph. If G is triangle-free and satisfies both conditions (C1) and (C2), then $PC_p(G)$ is integral, for any integer p.*

Also in [2] we proved that conditions (C1) and (C2) are necessary. Now, to complete the proof of Theorem 2, we need to prove the following.

Theorem 5. *Let $G = (V, A)$ be an oriented graph containing a non-directed g-odd Y-cycle and satisfying (C1) and (C2), then $PC_p(G)$ is integral, for any integer p.*

The proof of this theorem is given in Section 3, it uses Theorem 4 as the starting point of the induction on the number of triangles in G.

In Section 2 we give some properties of the extreme points of $PC_p(G)$ in general graphs, this will be used for our proof. We finish this section giving some definitions that will be used extensively, and a previous result on the p-median polytope that will also be used.

For $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of arcs (u, v), with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively.

Definition 6. *A node that has one arc entering it, and no arc leaving it is called a pendent node.*

Definition 7. *The graph $G(s, t)$ is obtained from G by removing the arc (s, t) and adding the arc (s, t') with t' a new pendent node.*

Definition 8. *When dealing with a vector $z \in PC_p(G)$, we say that the arc (u, v) is tight if*

$$z(u, v) = z(v).$$

In [1] we characterized the oriented graphs for which $P_p(G)$ is integral, as follows.

Theorem 9. *Let $G = (V, A)$ be an oriented graph, then $P_p(G)$ is integral if and only if*

- *(i) it does not contain as a subgraph any of the graphs F_1, F_2 or F_3 of Figure 2, and*
- *(ii) it does not contain a g-odd Y-cycle C and an arc (u, v) with neither u nor v in $V(C)$.*

Notice that condition (ii) forbids an odd directed cycle C with an extra arc whose endnodes are not in C, whereas condition (C2) of Theorem 2 allows this configuration.

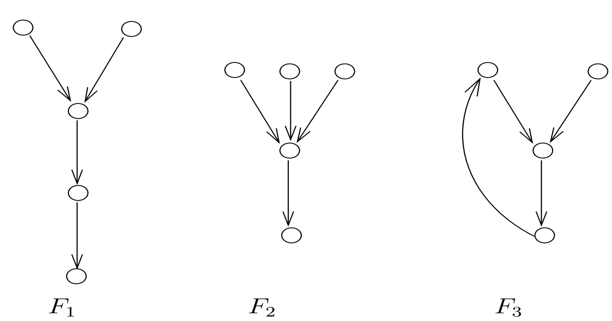

FIGURE 2

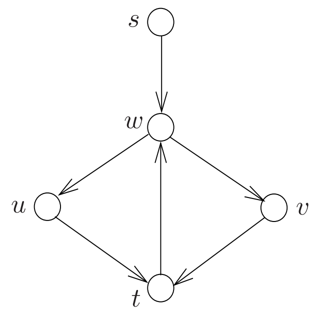

FIGURE 3. The graph H.

2. Some useful properties of the extreme points of $PC_p(G)$

Here we present some operations on the graph, and we discuss their consequences on the polytope.

Let H be the graph of Figure 3. Let G_1 be any directed graph and define G to be the graph obtained by composing H and G_1 as follows: pick any two distinct nodes in G_1, call them u' and v' and join the graphs G_1 and H by adding the arcs (u', u) and (v', v), see Figure 4.

Define G_1' to be the graph defined from G by removing the node w and its incident arcs and by adding the arc (t, w'), where w' is a new pendent node. Define G_1'' to be the graph obtained from G_1' by adding an arc (s', t') where both s' and t' are new nodes, so t' is a pendent node. The graphs G_1' and G_1'' are represented in Figure 4.

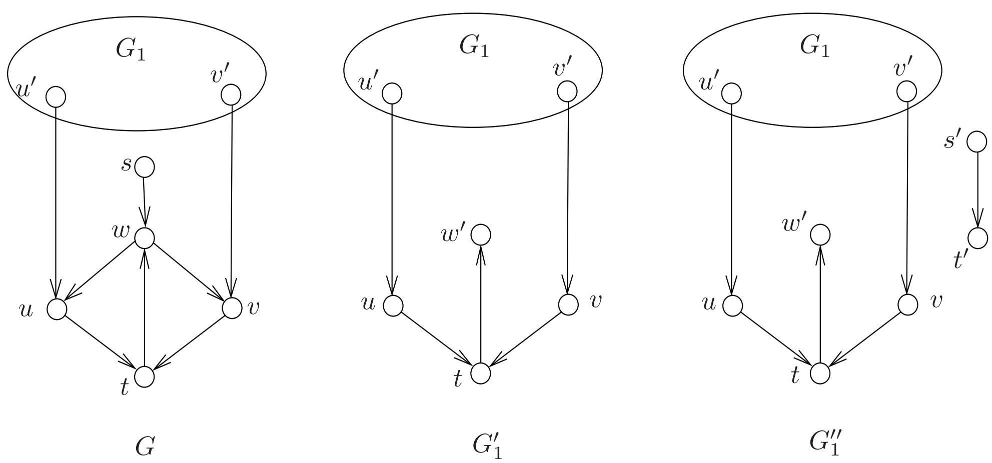

FIGURE 4. The graphs G, G_1' and G_1''.

Lemma 10. *If $PC_p(G_1')$ and $PC_{p+1}(G_1'')$ are integral, then $PC_p(G)$ does not have a fractional extreme point z, with $z(t, w) > 0$.*

Proof. Assume that $PC_p(G_1')$ and $PC_{p+1}(G_1'')$ are integral. Let z be such a fractional extreme point of $PC_p(G)$, with $z(t, w) > 0$. We distinguish two cases, the arc (s, w) is tight or not. Before the study of these cases let us notice two useful facts.

Remark 11. *By the definition of G any odd directed cycle is completely included in G_1 or in H.*

Remark 12. *The definition of H and the validity of z imply that the arcs (w,v) and (w,u) are not tight.*

Proof. Assume that (w,v) is tight. Then $z(w,v) = z(v) = 1 - z(v,t)$. But then the odd cycle inequality $z(w,v) + z(v,t) + z(t,w) \leq 1$ implies $z(t,w) = 0$.

The proof in the other case is similar. □

Case 1. $z(s,w) = z(w)$.

Define z'_1 to be the restriction of z on G'_1 with $z'_1(t,w') = z(t,w)$ and $z'_1(w') = 1$. Notice that the fact that $z(s,w) = z(w)$ and the validity of z imply that $z(s) + z(w) = 1$. Therefore, by definition $z'_1 \in PC_p(G'_1)$. Since by hypothesis, $PC_p(G'_1)$ is integral this implies that z'_1 may be written as a convex combination of 0-1 vectors in $PC_p(G'_1)$ that satisfy with equation each constraint among (1)-(6) that is satisfied with equation by z'_1. Among these solutions choose one, z^*, with $z^*(t) = 0$. Notice that, $z^*(u) = z^*(v) = 1$.

Define the solution z_1^H as follows: $z_1^H(t,w) = z_1^H(s,w) = z_1^H(w) = 1$; $z_1^H(u) = z_1^H(v) = 1$; and $z_1^H(\cdot) = 0$ for all other nodes and arcs of H.

From z_1^H and z^* we will define $\bar{z} \in PC_p(G)$ as follows. Every arc and node that belongs to H and G takes the value of z_1^H; otherwise it takes the value of z^*. Formally,

$$\bar{z}(r) = \begin{cases} z_1^H(r) & \text{if } r \in V(H), \\ z^*(r) & \text{if } r \in V(G_1), \end{cases} \qquad \bar{z}(r,q) = \begin{cases} z_1^H(r,q) & \text{if } (r,q) \in E(H), \\ z^*(r,q) & \text{if } (r,q) \in E(G_1), \\ z^*(u',u) & \text{if } (r,q) = (u',u), \\ z^*(v',v) & \text{if } (r,q) = (v',v), \end{cases}$$

It is easy to check that $\bar{z} \in PC_p(G)$. Remarks 11 and 12 help to establish that every constraint in $PC_p(G)$ that is satisfied with equation by z is also satisfied with equation by $\bar{z}$. This is a contradiction with the fact that z is an extreme point of $PC_p(G)$.

Case 2. $z(s,w) < z(w)$.

Let $\delta = z(w) - z(s,w)$. The validity of z imply that $z(s) + z(w) = 1 + \delta$. The solution z'_1 as defined in Case 1 does not belong to any polytope $PC_n(G'_1)$ with n integer, since $\sum z'_1(v) = p - \delta$ is fractional. Hence we need another valid transformation which is the graph G''_1. Define z''_1 to be the restriction of z on G''_1 with $z''_1(t,w') = z(t,w)$; $z''_1(w') = 1$; $z''_1(s') = \delta$; $z''_1(s',t') = 1 - \delta$ and $z''_1(t') = 1$. It is clear that $z''_1 \in PC_{p+1}(G''_1)$. Therefore, since $PC_{p+1}(G''_1)$ is integral the solution z''_1 may be written as a convex combination of 0-1 solutions of $PC_{p+1}(G''_1)$ that satisfy with equation each constraint among (1)-(6) that is satisfied with equation by z''_1. Among these solutions choose the one, where the variable associated with t is zero. Call this solution z^*. Notice that if $z^*(s') = 0$, then the solution $\bar{z}$ as defined in Case 1 belongs to $PC_p(G)$ and yields to the same contradiction as above. Let us assume that $z^*(s') = 1$, in this case the solution $\bar{z}$ as defined above is no longer in $PC_p(G)$ but it belongs to $PC_{p-1}(G)$. To transform $\bar{z}$ to a solution in $PC_p(G)$ it is sufficient to transform z_1^H, the solution associated with the graph H and defined in Case 1. We define z_2^H as follows: $z_2^H(s,w) = 0$ and $z_2^H(s) = 1$; otherwise $z_2^H(\cdot) = z_1^H(\cdot)$. We define $\bar{z} \in PC_p(G)$ as in Case 1 where z_1^H is exchanged with z_2^H.

Here also we obtain the same contradiction as in the previous case. Notice that $\bar{z}(s,w) = z_2^H(s,w) = 0 < \bar{z}(w) = z_2^H(w) = 1$. But recall that we also have $z(s,w) < z(w)$. □

Let H' be the graph of Figure 3, but without the node s and the arc (s, w). Let G_1 be any directed graph and define G' to be the graph obtained by composing H' and G_1 as follows: take any two distinct nodes in G_1, call them u' and w' and join the graphs G_1 and H' by adding the arcs (u', u) and (w', w), see Figure 5.

Define G_1^* to be the graph defined from G' by removing the node v and t and their incident arcs and by adding the arcs (u, t'), (w, s) and (s, t''), where s, t' and t'' are new nodes and t' and t'' are pendent nodes. Define G_2^* to be the graph obtained from G' by removing w and t and their incident arcs and by adding the arcs (u, v), (w', v) and (v, t') where t' is a new pendent node. The graphs G_1^* and G_2^* are shown in Figure 5.

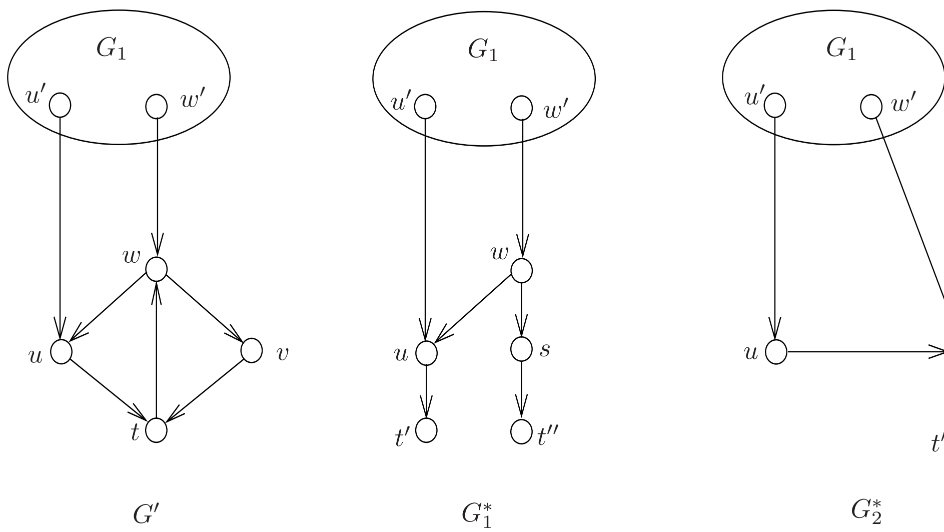

FIGURE 5. The graphs G', G_1^* and G_2^*.

Lemma 13. *If $PC_{p+1}(G_1^*)$ and $PC_p(G_2^*)$ are integral, then $PC_p(G)$ does not have a fractional extreme point z with $z(w, u) > 0$, $z(w, v) > 0$, $z(t, w) > 0$, $z(v, t) > 0$.*

Proof. Assume that $PC_{p+1}(G_1^*)$ and $PC_p(G_2^*)$ are integral but $PC_p(G)$ is not.

Let z be a fractional extreme point of $PC_p(G)$. First notice that the validity of z implies that the arcs (w, u), (u, t), and (w, v) are not tight. This fact will be used implicitly in the proof. Let $\Delta_1 = \{(w, v), (v, t), (t, w)\}$ and $\Delta_2 = \{(w, u), (u, t), (t, w)\}$. We need the two relations below.

- We have $z(\Delta_1) < 1$.

  To prove it assume that $z(\Delta_1) = z(w, v) + z(v, t) + z(t, w) = 1$. By the validity of z we have $z(v, t) = 1 - z(v)$ and $z(t, w) = 1 - z(t)$. Combining with $z(\Delta_1) = 1$, we obtain

$$z(v) + z(t) = 1 + z(w, v). \tag{7}$$

  Define from z a solution $z_1^* \in PC_{p+1}(G_1^*)$ as follows: $z_1^*(w, s) = z_1^*(s) = z(w, v)$, $z_1^*(s, t'') = 1 - z(w, v)$; $z^*(u, t') = z(u, t)$ and $z_1^*(t') = z_1^*(t'') = 1$; for all other arcs and nodes we set $z_1^*(\cdot) = z(\cdot)$. Using (7) and the definition of z_1^*, it is easy to check that $\sum z_1^*(v) = p + 1$. It is obvious to see that $z_1^* \in PC_{p+1}(G_1^*)$. Since $PC_{p+1}(G_1^*)$ is integral, we may choose a 0-1 solution in $PC_{p+1}(G_1^*)$, call it z^1 that satisfies with equation each constraint that is satisfied with equation by z_1^*, where $z^1(w,u) = 1$. From z^1 one can define $\bar{z} \in PC_p(G')$ as follows: $\bar{z}(w) = \bar{z}(t,w) = \bar{z}(w,v) = \bar{z}(v) = 0$; $\bar{z}(v,t) = \bar{z}(t) = 1$; $\bar{z}(u,t) = 0$ and otherwise $\bar{z}(\cdot) = z^1(\cdot)$. It is obvious to see that $\bar{z}$ contradicts the fact that z is an extreme point of $PC_p(G'$.

- We have $z(\Delta_2) = 1$.

  Otherwise, from the claim above we have $z(\Delta_1) < 1$, and the solution obtained from z by adding a sufficiently small positive scalar ϵ to $z(w,v)$ and removing this same value from $z(w,u)$ contradicts the fact that z is extreme.
- We should have $z(t,w) = z(w)$.

  Otherwise the solution obtained from z by adding a some positive scalar ϵ to $z(w,u)$, $z(v,t)$ and to $z(t)$ and removing this same value from $z(w,v)$, $z(v)$ and $z(t,w)$; we obtain a solution that satisfies with equation each constraint that z satisfies with equation.

Hence we may assume that

$$z(\Delta_1) < 1, \tag{8}$$

$$z(\Delta_2) = 1, \tag{9}$$

$$z(t,w) = z(w). \tag{10}$$

Combining $z(t,w) = z(w)$ and $z(t,w) = 1 - z(t)$, we obtain

$$z(w) + z(t) = 1. \tag{11}$$

From (9) we have $z(w,u) + z(u,t) + z(t,w) = 1$. The validity of z implies that $z(w,u) = 1 - z(w) - z(w,v)$. Combining these two last equations with (10) we obtain

$$z(w,v) = z(u,t). \tag{12}$$

From (8) and (10), we have $z(w) + z(w,v) < z(v)$. Therefore, we may conclude that

$$z(w) < z(v). \tag{13}$$

Let us define $z_2^* \in PC_p(G_2^*)$ from z as follows: $z_2^*(u,v) = z(u,t)$; $z_2^*(w',v) = z(w',w)$; $z^*(v,t') = z(v,t)$ and $z_2^*(t') = 1$; otherwise $z_2^*(\cdot) = z(\cdot)$.

By definition we have $\sum z_2^*(v) = p$. We also have by the definition of z_2^*, the validity of z and (12) and (13), that $z_2^*(u,v) = z(u,t) = z(w,v) \leq z(v) = z_2^*(v)$ and $z_2^*(w',v) = z(w',w) \leq z(w) < z(v) = z_2^*(v)$. The validity of the other constraints is straightforward.

Let z_1 be a 0-1 solution of $PC_p(G_2^*)$ with $z_1(v) = 0$ that satisfies with equation each constraint that z_2^* satisfies with equation.

Since $z_1(v) = 0$, its validity imply that $z_1(w',v) = 0$; $z_1(u,v) = 0$; $z_1(u) = 1$; $z_1(v,t') = 1$.

Then $\bar{z} \in PC_p(G')$ is defined as follows:

$$\begin{aligned}
&\bar{z}(u) = \bar{z}(w,u) = \bar{z}(t) = \bar{z}(v,t) = 1,\\
&\bar{z}(w',w) = \bar{z}(t,w) = \bar{z}(w) = \bar{z}(u,t) = \bar{z}(v) = \bar{z}(w,v) = 0,\\
&\bar{z}(\cdot) = z_1(\cdot) \text{ for all other arcs and nodes.}
\end{aligned}$$

It is a simple exercise to check that $\bar{z} \in PC_p(G')$ and that it satisfies with equation each constraint that is satisfied with equation by z. □

Lemma 14. *If the following statements hold*

(i) *z is a fractional extreme point of $PC_p(G)$,*
(ii) *The only inequality of $PC_p(G)$ that z satisfies with equation is $z(s,t) - z(t) \leq 0$, and*
(iii) *$PC_{p+1}(G(s,t))$ is integral,*

then $z(s,t) = z(t) = \frac{1}{2}$.

Proof. Assume that (i), (ii) and (iii) hold.

Let z' be the restriction of z on $G(s,t)$ with $z'(s,t') = z(s,t)$ and $z'(t') = 1$. Clearly $z' \in PC_{p+1}(G(s,t))$. Since $PC_{p+1}(G(s,t))$ is integral and since z' is fractional, z' may be written as convex combination of 0-1 solutions of $PC_{p+1}(G(s,t))$, $z^1, \ldots, z^l$, where each constraint of $PC_{p+1}(G(s,t))$ that is satisfied with equation by z' is also satisfied with equation by each solution z^i, $i = 1, \ldots, l$. We have

$$z' = \sum_{i=1}^{l} \lambda^i z^i, \tag{14}$$

$$\sum_{i=1}^{l} \lambda^i = 1, \tag{15}$$

$$\lambda^i \geq 0, \quad i = 1, \ldots, l. \tag{16}$$

Assume first that there exist one solution among $z^1, \ldots, z^l$, say z^1 with $z^1(s,t') = z^1(t)$. Define $\bar{z}$ from z^1 as follows: $\bar{z}(\cdot) = z^1(\cdot)$ except for the arc (s,t) we set $\bar{z}(s,t) = z^1(s,t') = z^1(t)$. The solution $\bar{z}$ as defined belongs to $PC_p(G)$. Moreover, it satisfies with equation each constraint that is satisfied with equation by z, this is due to (ii). This contradicts the fact that z is an extreme point of $PC_p(G)$. Thus in each solution z^i for $i = 1, \ldots, l$, we have $z^i(s,t') \neq z^i(t)$. From the system (14)-(16), there exist $0 \leq l_1 \leq l$, such that

$$\sum_{i=1}^{l_1} \lambda^i = z'(s,t'), \tag{17}$$

$$\sum_{i=l_1+1}^{l} \lambda^i = z'(t). \tag{18}$$

Since $z'(s,t)$ is fractional we have that $l_1 < l$. And since by definition $z'(s,t') = z(s,t)$, $z'(t) = z(t)$ and that from (ii) $z(s,t) = z(t)$ so $z'(s,t') = z'(t)$. This combined with (17), (18) and (15) imply that $z'(s,t') = z'(t) = \frac{1}{2}$. Therefore we have

$$z(s,t) = z(t) = \frac{1}{2}.$$

□

3. Proof of Theorem 5

Here we use induction on the number of triangles to prove Theorem 5. Recall that if there is no non-directed g-odd Y-cycle, the result follows from Theorem 3, and if there is no triangle it follows from Theorem 4.

We assume that the oriented graph $G = (V, A)$ satisfies conditions (C1) and (C2) of Theorem 2, it contains a non-directed g-odd Y-cycle, and that z is a fractional extreme point of $PC_p(G)$. Based on this we plan to arrive to a contradiction. In [2], Lemmas 14-16, we showed that arcs (u, v) with $z(u, v) = 0$ can be removed, and nodes v with $\delta^+(v) = \emptyset$ can be splitted. Thus we assume that G and z have the following properties:

- $z(u, v) > 0$ for all $(u, v) \in A$,
- $z(v) > 0$ for all $v \in V$ with $|\delta^-(v)| \geq 1$, and
- $|\delta^-(v)| = 1$, for every node v with $\delta^+(v) = \emptyset$, i.e., v is a pendent node.

These properties will be used implicitly throughout the paper.

Let $\Delta(G)$ be the set of triangles in G. We assume that $|\Delta(G)| = m + 1$ and that $PC_p(G')$ is integral for any graph G' satisfying (C1) and (C2), with $|\Delta(G')| \leq m$, and for any value p. This is our induction hypothesis.

We start with some Lemmas.

3.1. **Some useful lemmas.**

Lemma 15. *Let (s, t) be an arc that does not belong to any odd cycle such that its associated inequality is tight. If $PC_{p+1}(G(s, t))$ is integral, then $z(s, t) = z(t)$.*

Proof. Assume that $z(s, t) < z(t)$. We define $z' \in PC_{p+1}(G(s, t))$, where $z'(s, t') = z(s, t)$, $z'(t') = 1$, $z'() = z()$ for all other nodes and arcs. Since $PC_{p+1}(G(s, t))$ is integral, z' can be written as convex combination of $0-1$ vectors in $PC_{p+1}(G(s, t))$ that satisffy with equation every constraint of $PC_{p+1}(G(s, t))$ that is satisffied with equation by z'. Among these vectors there is one, $\bar{z}$, with $\bar{z}(s, t') = 0$.

Then we define $\tilde{z} \in PC_p(G)$, where $\tilde{z}(s, t) = \bar{z}(s, t')$, $\tilde{z}() = \bar{z}()$ for all other nodes and arcs. With the exception of $z(s, t) \leq z(t)$, $\tilde{z}$ satisfies with equation all other constraints of $PC_p(G)$ that z satisfies with equation. Since z is an extreme point, the only possibility is to have $\tilde{z}(s, t) < \tilde{z}(t)$ and $z(s, t) = z(t)$. ☐

Lemma 16. *Let (s, t) be an arc of a triangle $\Delta = \{(s, t), (t, w), (w, s)\}$ in G, with $z(s, t) < z(t)$. If (s, t) does not belong to any odd directed cycle different from Δ, whose associated inequality is satisffied with equation, then $PC_{p+1}(G(s, t))$ is not integral.*

Proof. Assume that $PC_{p+1}(G(s, t))$ is an integral polytope. Let z' the solution defined from z as follows: $z'(u, v) = z(u, v)$ for all $(u, v) \in A$, different from (s, t') and $z'(s, t') = z(s, t)$; $z'(u) = z(u)$ for all $u \in V$ and $z'(t') = 1$. It is clear that $z' \in PC_{p+1}(G(s, t))$.

Since $PC_{p+1}(G(s, t))$ is integral, z' is a convex combination of 0-1 vectors in $PC_{p+1}(G(s, t))$. Among them there is a vector, call it z^*, with $z^*(w, s) = 1$. This imply that $z^*(s) = 1$; $z^*(s, t') = 0$; $z^*(w) = 0$, $z^*(t, w) = 0$.

From z^* construct a solution $\bar{z}$ as follows: $\bar{z}(u, v) = z^*(u, v)$ for all $(u, v) \in A$, $(u, v) \neq (s, t)$; $\bar{z}(s, t) = 0$; and $\bar{z}(u) = z^*(u)$ for each $u \in V$. Notice that $\bar{z}(\Delta) = 1$, and since (s, t) does not belong to any other odd directed cycle whose associated inequality is tight, each inequality tight for z remains tight for $\bar{z}$. This contradicts the fact that z is an extreme point of $PC_p(G)$. □

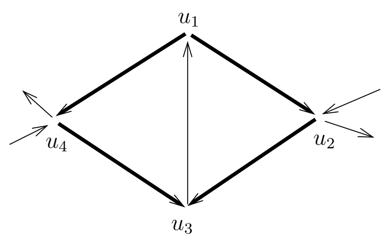

FIGURE 6. A cycle of size four in bold with $\delta^+(u_3) = \delta^-(u_1) = \{(u_3, u_1)\}$.

Lemma 17. *Assume that G contains the cycle*

$$C = u_1, (u_1, u_2), u_2, (u_2, u_3), u_3, (u_4, u_3), u_4, (u_1, u_4), u_1,$$

with $\delta^+(u_3) = \delta^-(u_1) = \{(u_3, u_1)\}$, see Figure 6. Then both graphs $G(u_2, u_3)$ and $G(u_4, u_3)$ satisfy conditions (C1) and (C2).

Proof. We give the proof for $G(u_2, u_3)$, the other proof is similar. It is straightforward that $G(u_2, u_3)$ does not contain H_2. And since $\delta^-(u_1) = \delta^+(u_3) = \{(u_3, u_1)\}$, then the configuration H_1 cannot occur, otherwise it is also present in G. Also if H_3 is present, it would also be present in G.

Now suppose that (C2) is not satisfied. Let C' be a non-directed g-odd Y-cycle in $G(u_2, u_3)$, with both nodes u_2 and u'_3 not in $V(C')$. This cycle is also a non-directed g-odd Y-cycle in G. But since G satisfies (C2), the node u_3 must belong to $V(C')$ and by definition (u_4, u_3) and (u_3, u_1) belong to $A(C')$. Then (u_1, u_4) also belongs to C' and we obtain the triangle induced by $\{u_1, u_3, u_4\}$. □

Lemma 18. *There is no intersection between two odd directed cycles having both size at least five.*

Proof. Assume that C_1 and C_2 are such cycles. Suppose that $(u, v) \in C_1 \setminus C_2$, $(w, v) \in C_2 \setminus C_1$, and $(v, t) \in C_1 \cap C_2$. Then in order to not have H_1, we should have an arc $(t, u) \in C_2 \setminus C_1$ and an arc $(t, w) \in C_1 \setminus C_2$.

With the same argument, in order to not have H_1, we obtain blocks defined as follows. For $k = 3j + 1$, $j = 0, \ldots, 2p$, each block B_j contains

- the nodes $v_k, v_{k+1}, v_{k+2}, v_{k+3}$, and
- the arcs $(v_k, v_{k+1}) \in C_1$, $(v_{k+1}, v_{k+2}) \in C_1 \cap C_2$, $(v_{k+2}, v_{k+3}) \in C_1$, $(v_{k+2}, v_k) \in C_2$, $(v_{k+3}, v_{k+1}) \in C_2$,

with $v_{6p+4} = v_1$.

Because of (C1), the only other arcs that the graph could contain are (v_k, z), with $z \notin C_1 \cup C_2$, in each block B_j. See Figure 7.

For each block B_j we have the triangle $T_j = \{(v_k, v_{k+1}), (v_{k+1}, v_{k+2}), (v_{k+2}, v_k)\}$ and $T'_j = \{(v_{k+1}, v_{k+2}), (v_{k+2}, v_{k+3}), (v_{k+3}, v_{k+1})\}$.

If we add the inequalities associated with C_1 and C_2 we obtain

$$z(C_1) + z(C_2) \leq 6p + 2.$$

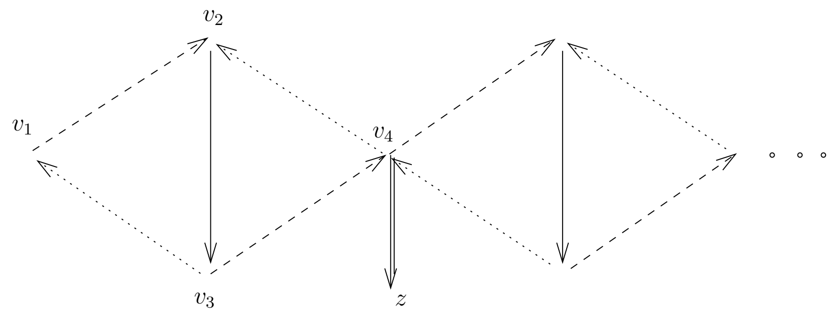

FIGURE 7. The cycles C_1 and C_2. The arcs in $C_1 \setminus C_2$ appear in dashed lines. The arcs in $C_2 \setminus C_1$ appear with dotted lines. The arcs in $C_1 \cap C_2$ appear in solid lines. Arcs not in $C_1 \cup C_2$ appear with double lines.

Adding the inequalities associated with the triangles we have

$$z(C_1) + z(C_2) = \sum_j \left(z(T_j) + z(T'_j) \right) \leq 4p + 2.$$

Therefore we cannot have $z(C_1) = 3p + 1$ and $z(C_2) = 3p + 1$ at the same time.

So assume that $z(C_1) < 3p + 1$. We also have $z(v_1, v_2) < z(v_2)$, otherwise we would have $z(v_3, v_1) = 0$. Now we build $G(v_1, v_2)$. Let z' be the solution defined from z as follows:

- $z'(u, v) = z(u, v)$ for all $(u, v) \in A$, $(u, v) \neq (v_1, v_2)$, and $z'(v_1, v'_2) = z(v_1, v_2)$;
- $z'(u) = z(u)$ for all $u \in V$ and $z'(v'_2) = 1$.

It is clear that $z' \in PC_{p+1}(G(v_1, v_2))$. From Lemma 17 the graph $G(v_1, v_2)$ satisfy conditions (C1) and (C2), so the induction hypothesis applies and we have that $PC_{p+1}(G(v_1, v_2))$ is integral. Thus z' is a convex combination of $0-1$ vectors in $PC_{p+1}(G(v_1, v_2))$, and among them there is a vector z^*, with $z^*(v_3, v_1) = 1$. This implies that $z^*(v_1) = 1$; $z^*(v_1, v'_2) = 0$; $z^*(v_3) = 0$, $z^*(v_2, v_3) = 0$; $z^*(v_2) = 1$. See Figure 8.

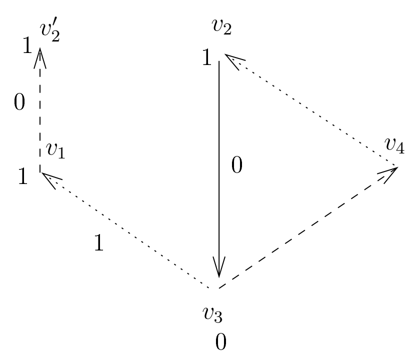

FIGURE 8

From z^* construct a solution $\bar{z}$ as follows:

- $\bar{z}(u, v) = z^*(u, v)$ for all $(u, v) \in A$, $(u, v) \neq (v_1, v_2)$; $\bar{z}(v_1, v_2) = 0$;
- and $\bar{z}(u) = z^*(u)$ for each $u \in V$.

Notice that $\bar{z}(v_1, v_2) + \bar{z}(v_2, v_3) + \bar{z}(v_3, v_1) = 1$ and since (v_1, v_2) does not belongs to any other directed cycle whose associated inequality is tight, then any tight odd directed cycle inequality for z remains tight for $\bar{z}$. Therefore, each inequality tight for z remains tight for $\bar{z}$, which contradicts the fact that z is an extreme point of $PC_p(G)$. □

Lemma 19. *G does not contain a cycle*

$$C = u_1, (u_1, u_2), u_2, (u_2, u_3), u_3, (u_4, u_3), u_4, (u_1, u_4), u_1,$$

where $\delta^+(u_3) = \delta^-(u_1) = \{(u_3, u_1)\}$. See Figure 6.

Proof. Assume that G contains the cycle C. We will prove that both the induction hypothesis and that of Lemma 16 hold for $G(u_2, u_3)$ or $G(u_4, u_3)$, which is a contradiction.

Notice that the validity of z imply

$$z(u_2, u_3) < z(u_3), \tag{19}$$

$$z(u_4, u_3) < z(u_3). \tag{20}$$

It is clear that at least one of the arcs (u_2, u_3) or (u_4, u_3) does not belong to any other odd directed cycle different from the triangles $\Delta_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\}$ and $\Delta_2 = \{(u_1, u_4), (u_4, u_3), (u_3, u_1)\}$. Otherwise, they belong to two different odd directed cycle of size at least five that have the common arc (u_3, u_1), but this contradicts Lemma 18.

As a consequence, Lemma 16 applies for $G(u_2, u_3)$ or for $G(u_4, u_3)$. Hence

$$PC_{p+1}(G(u_2, u_3)) \text{ or } PC_{p+1}(G(u_4, u_3)) \text{ is not integral .} \tag{21}$$

From Lemma 17 the graphs $G(u_2, u_3)$ and $G(u_4, u_3)$ satisfy conditions (C1) and (C2). Therefore, the induction hypothesis applies for both graphs $G(u_2, u_3)$ and $G(u_4, u_3)$, which implies that $PC_{p+1}(G(u_2, u_3))$ and $PC_{p+1}(G(u_4, u_3))$ are integral. This contradicts (21). □

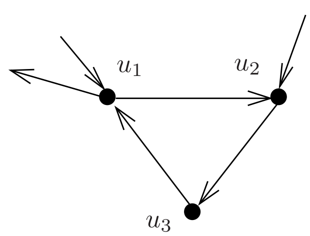

FIGURE 9. A triangle with $\delta^+(u_3) = \{(u_3, u_1)\}$; $\delta^-(u_3) = \{(u_2, u_3)\}$ and $\delta^+(u_2) = \{u_2, u_3)\}$

Lemma 20. *G does not contain a triangle $\Delta = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\}$, with $\delta^+(u_3) = \{(u_3, u_1)\}$, $\delta^-(u_3) = \{(u_2, u_3)\}$ and $\delta^+(u_2) = \{u_2, u_3)\}$, see Figure 9.*

Proof. By hypothesis, we have $\delta^+(u_2) = \{(u_2, u_3)\}$. Combining this with the validity of z we obtain that $z(u_1, u_2) < z(u_2)$. Notice that the only odd directed cycle that contains (u_1, u_2) is Δ. So Lemma 16 applies for Δ and the arc (u_1, u_2) and it implies that $PC_{p+1}(G(u_1, u_2))$ is not integral. By the lemma hypothesis it is easy to establish that $G(u_1, u_2)$ satisfies condition (C1). Assume that $G(u_1, u_2)$ contains a g-odd Y-cycle C' such that both nodes u_1 and u'_2 are not in C'. So C' is also a g-odd Y-cycle in G and since G satisfies condition (C2), u_2 must be in $V(C')$ but not u_1. Because of (C2), u_3 should be in C' that is impossible.

Thus the induction hypothesis applies, and we have a contradiction, since we have shown above that $PC_{p+1}(G(u_1, u_2))$ is not integral. □

Lemma 21. *Let $\Delta = \{(u, v), (v, w), (w, u)\}$. If $\delta^-(\{u, v, w\}) = \emptyset$, and u, v and w have a neighbor in $V \setminus \{u, v, w\}$ and none in common, then $G(s, t)$ satisfies conditions (C1) and (C2) for any arc $(s, t) \in \Delta$.*

Proof. Let $(s, t) = (u, v)$. It is easy to see that $G(u, v)$ satisfies (C1). Now we have to see that $G(u, v)$ satisfies (C2).

Suppose that C is a non-directed g-odd Y-cycle in $G(u, v)$, and that u and v' are not in $V(C)$. Since G satisfies (C2), the nodes v and w should be in $V(C)$. We have the two cases below.

Case 1: $v, w \in \dot{C}$. Let P_1 and P_2 be the two paths in C between v and w. One of these two paths, P_1 say, together with (v, w) form a g-odd Y-cycle. Condition (C2) implies that P_2 should consist of exactly two arcs. Let (w, t) and (v, t) be these two arcs, we have a contradiction because v and w should not have a common neighbor different from u.

Case 2: $v \in \dot{C}$ and $w \in \tilde{C}$. Let $\bar{v}$ and $\bar{w}$ the neighbors of v and w, respectively, in C. Also recall that there is an arc $(u, \bar{u})$ with $\bar{u} \notin \{u, v, w\}$. The node $\bar{u}$ must be in $V(C)$, otherwise C with $(u, \bar{u})$ violates (C2) in G. Let P_1 be the path in C between $\bar{w}$ and $\bar{u}$ not containing $\bar{v}$, and P_2 the path between $\bar{u}$ and $\bar{v}$ not containing $\bar{w}$.

Recall from the lemma hypothesis $\bar{u}$, $\bar{v}$ and $\bar{w}$ are different. Let C_1 the cycle obtained by joining P_1 with $(u, \bar{u}), (w, u)$ and $(w, \bar{w})$. This is a Y-cycle, it must be g-even otherwise C_1 and $(v, \bar{v})$ would violate (C2) in G. It implies that the cycle C_2 obtained from C_1 by removing (w, u) and adding (u, v) and (v, w) is a g-odd Y-cycle. Then the path P_2 consists of one arc, otherwise (C2) is violated. If P_2 consists of the arc $(\bar{v}, \bar{u})$ then the cycle induced by $\{u, \bar{u}, v, \bar{v}\}$ is a g-odd Y-cycle and with $(w, \bar{w})$ it violates (C2). If the arc $(\bar{u}, \bar{v})$ exists, we obtain the same violation of (C2). □

Lemma 22. *Let $\Delta = \{(u, v), (v, w), (w, u)\}$. If $\delta^-(\{u, v, w\}) = \emptyset$, and u, v and w have no neighbor in common, then at least one of the arcs of Δ is not tight.*

Proof. Assume that all arcs in Δ are tight. Notice that in this case each node of Δ has a neighbor outside Δ. We have the two cases below.

Case 1. $z(\Delta) < 1$. Take any arc of Δ, say (u, v), and consider the graph $G(u, v)$. Let z' be the restriction of z on $G(u, v)$ with $z'(u, v') = z(u, v)$ and $z'(v') = 1$. Clearly $z' \in PC_{p+1}(G(u, v))$. Since from Lemma 21, $G(u, v)$ satisfies (C1) and (C2) and has less triangles than G, then from the induction hypothesis we have that $PC_{p+1}(G(u, v))$ is integral. Notice that Lemma 14 applies with $(s, t) = (u, v)$. Statement (i) holds. To see that (ii) holds, notice that the arc (u, v) belongs to no other odd directed cycle other than Δ, and that $z(\Delta) < 1$. So $z(u, v) = z(v)$ is the unique tight inequality in $PC_p(G)$ that contains the variable $z(u, v)$. Statement (iii) holds by the induction hypothesis. Therefore, Lemma 14 implies that

$$z(u, v) = z(v) = \frac{1}{2}.$$

By symmetry we have the same for each arc of Δ. This implies that $z(\Delta) = \frac{3}{2}$, which is impossible.

Case 2. $z(\Delta) = 1$. This implies $z(\delta^+(\{u, v, w\})) = 1$. Here we build a graph G' by removing $\{u, v, w\}$, adding the node s, and the arcs (s, j) for each arc $(i, j) \in \delta^+(\{u, v, w\})$. The graph G' is a simple oriented graph, and it is easy to see that it satisfies (C1) and (C2). We define z' below.

- $z'(i, j) = z(i, j)$ for all $(i, j) \in A$, $(i, j) \notin \delta^+(\{u, v, w\}) \cup \Delta$.
- $z'(s, j) = z(i, j)$ for all $(i, j) \in \delta^+(\{u, v, w\})$.
- $z'(i) = z(i)$ for all $i \in V \setminus \Delta$ and $z'(s) = 0$.

We have that $z' \in PC_{p-1}(G')$. From the induction hypothesis we have that $PC_{p-1}(G')$ is integral, therefore z' is a convex combination of integral extreme points that satisfy with equation the same constraints that z' does. Let z^* be one of these vectors. We define $\bar{z} \in PC_p(G)$ as follows.

- $\bar{z}(i, j) = z^*(i, j)$ for all $(i, j) \in A$, $(i, j) \notin \delta^+(\{u, v, w\}) \cup \Delta$.
- $\bar{z}(i, j) = z^*(s, j)$ for all $(i, j) \in \delta^+(\{u, v, w\})$.
- $\bar{z}(i) = z^*(i)$ for all $i \in V \setminus \Delta$.
- Suppose that $z^*(s, j) = 1$ and (s, j) is associated with (u, j). Then we set $\bar{z}(u) = \bar{z}(v) = 0$, $\bar{z}(w) = 1$, $\bar{z}(v, w) = 1$, $\bar{z}(u, v) = \bar{z}(w, u) = 0$.

Then $\bar{z} \in PC_p(G)$ and it satisfies with equation all constraints that are tight for z, a contradiction. □

3.2. **Intersection of a triangle with a g-odd Y-cycle.** Now we assume that G contains a non-directed g-odd Y-cycle C. Recall that G satisfies conditions (C1) and (C2) and that z is an extreme fractional point of $PC_p(G)$. Also we assume that

- $z(u, v) > 0$ for all $(u, v) \in A$,
- $z(v) > 0$ for all $v \in V$ with $|\delta^-(v)| \geq 1$, and
- $|\delta^-(v)| = 1$, for every node v with $\delta^+(v) = \emptyset$, i.e., v is a pendent node.

Let $\Delta = \{(u, v), (v, w), (w, u)\}$ be a triangle, condition (C2) implies that at least two nodes of Δ are in C. Furthermore, it follows from Lemmas 27, 28 and 29 in the Appendix, that the configurations to be studied are the following.

- Only the arc (u, v) is in C, and w is not in C.
- The arcs (v, w) and (w, v) are in C.

They appear in Figures 10. Let u' (resp. v') be the neighbors in C of u (resp. v) that are different from w.

Lemma 23. *In case (c1) we have*

(i) $\delta^+(w) = \{(w, u)\}$,
(ii) $\delta^-(u) = \{(w, u)\}$,
(iii) $\delta^-(w) = \{(v, w)\}$,
(iv) *the only common neighbor of u and v is w.*

Proof. (i) Let $(w, \bar{u})$ be an arc in A with $\bar{u} \neq u$. To avoid H_1 we should have $\bar{u} = v'$. Also we should have $\delta^+(v) = \{(v, w)\}$, otherwise we would have H_3. If there is an arc $(\bar{w}, w)$, $\bar{w} \neq v$, we must have $\bar{w} = u'$. But then we also have H_3. Thus $\delta^-(w) = \{(v, w)\}$. Then we have a contradiction with Lemma 19.

(ii) Assume $\delta^-(u) \neq \{(w, u)\}$. Let $(\bar{u}, u) \in A$, with $\bar{u} \neq w$. Assume that there is an arc $(v, \bar{v})$ with $\bar{v} \neq w$. To avoid H_1 we must have $\bar{v} = \bar{u}$. Then the arcs $(v, \bar{u}), (\bar{u}, u), (u, u'), (u, v), (w, u)$ and (v, w) induce H_3.

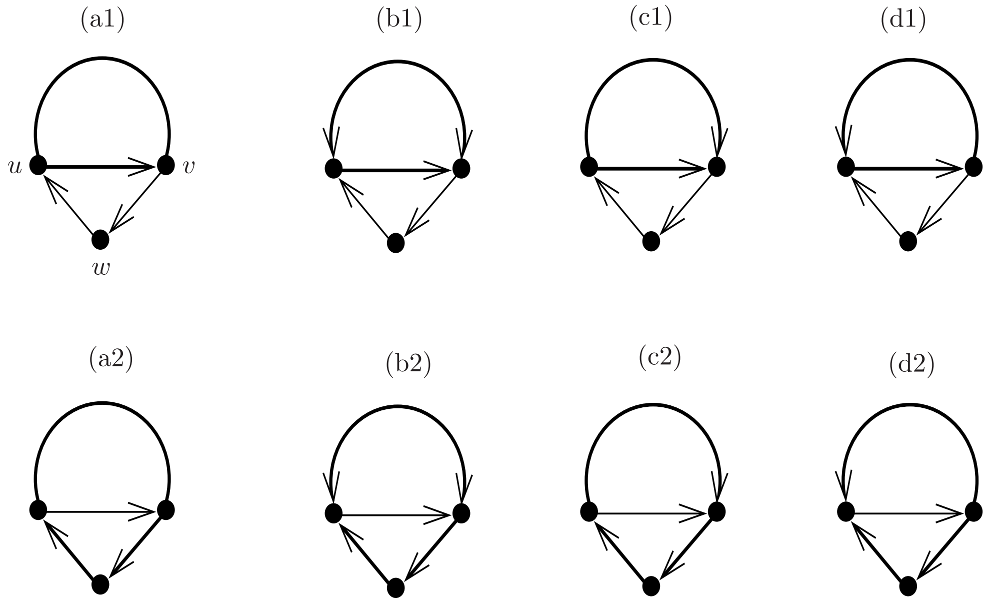

FIGURE 10. The cycle C is in bold. The different cases depend upon u and v are in $\dot{C}$, $\tilde{C}$, or $\hat{C}$.

Hence we may assume that $\delta^+(v) = \{(v,w)\}$. If $\delta^-(w) = \{(v,w)\}$, then Lemma 20 applies to $\Delta = \{(u,v),(v,w),(w,u)\}$. So we may assume that there is an arc $(\bar{w},w)$, $\bar{w} \neq v$. To avoid H_1 we must have $\bar{w} = u'$. The arc $(u',\bar{u})$ cannot exist, otherwise the nodes $u',\bar{u},u,v,w$ induce H_3. Thus there is exactly one arc leaving u', (u',w), and exactly two arcs entering it (u,u') and (u'',u'), otherwise we have H_1 or H_2 is present. Both arcs (u,u') and (u'',u') belong to C. And from (i), $\delta^+(w) = \{(w,u)\}$. See Figure 11.

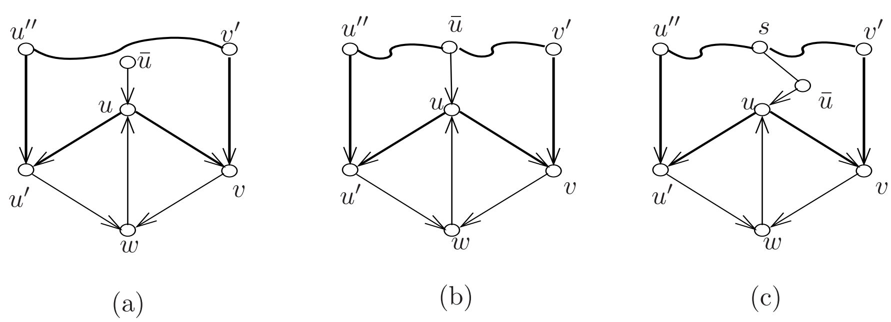

FIGURE 11. The cycle C is in bold. In all the cases the remaining arcs of G have both endnodes in $V \setminus \{u,v,w,u'\}$.

We treat the different possibilities of $\bar{u}$. By definition $\bar{u} \notin \{u,v,w,u'\}$. Also $u'' \neq v'$, otherwise C is g-even.

- $\bar{u} \in \{v',u''\}$. In any case we create a g-odd Y-cycle and (C2) implies that $\bar{u} = v' = u''$. But in this case C is g-even.
- $\bar{u} \in V(C)$. See Figure 11 (b). Define C_1 the cycle obtained by joining the path from $\bar{u}$ to v' with the arcs $(\bar{u},u)$, (u,v) and (v',v). Also define C_2 to be the cycle obtained from C_1 by removing (u,v) and adding (v,w) and (w,u).

Notice that these two cycles have different parity. Since C is a Y-cycle then they are both Y-cycles. Since $u'' \neq v'$, and $\bar{u} \neq u''$, the cycle C_1 or C_2 with the arc (u'', u') violate (C2), a contradiction.

– $\bar{u}$ is adjacent to a node s in C. If $s = u'$ we would have H_3. If $s = u''$, the nodes $u, u', u'', \bar{u}$, induce a non-directed g-odd Y-cycle, that together with (v, w) violate (C2). Thus the node s must be in the path of C joining u'' to v' that does not contain u, see Figure 11 (c). Now we consider the cycle C_1 that joins the path between s and v', the arc between s and $\bar{u}$ and the arcs $(\bar{u}, u)$, (u, v) and (v', v). And define C_2 to be the cycle defined from C_1 by removing (u, v) and adding (v, w) and (w, u). These two cycles have the same properties as the cycles of the previous case and lead to the same contradiction.

From this discussion we may assume that the unique neighbor of $\bar{u}$ is u as pictured in Figure 11 (a). We will apply Lemma 10. The graph H of this lemma induced by the nodes u, v, w, u' and $\bar{u}$. Also notice that G'_1 and G'_2 as constructed in Lemma 10 satisfy conditions (C1) and (C2). Condition (C1) is obviously satisfied and (C2) also since none of these two graphs contains a non-directed g-odd Y-cycle. Moreover both graphs have less triangles than G. Hence the induction hypothesis applies and implies that $PC_p(G'_1)$ and $PC_{p+1}(G'_2)$ are integral. Therefore Lemma 10 applies and implies that $z(w, u) = 0$, which is impossible.

(iii) As above if there is an arc $(\bar{w}, w)$, $\bar{w} \neq v$, we should have $\bar{w} = u'$. Based on (i) and (ii) we have that the cycle induced by $\{u', u, v, w\}$ contradicts Lemma 19.

(iv) Let t be a common neighbor of u and v, $t \neq w$. First we have to show that $t \neq v'$. Suppose that the arc (u, v') exists. We have two cases.

– If $v' \in \dot{C}$, then we can remove from C the arcs (u, v) and (v', v) and add (u, v'). This gives us a new non-directed g-odd Y-cycle that together with (v, w) violate (C2).

– If $v' \in \tilde{C}$ then the graph H_1 is present.

Now we assume that $t \neq v'$, and it follows from (C1) and (ii) that (v, t) and (u, t) are the only arcs that can make this possible. We have two cases.

– If $t \notin V(C)$, since t is not pendent, and because of (C2), there is an arc $(t, \bar{t})$ with $\bar{t} \in V(C)$. Since the cycle induced by $\{t, u, v, w\}$ is a g-odd Y-cycle, condition (C2) implies that C is of size three and g-even.

– If $t \in V(C)$, we need two remarks.

 * We should have $u' \neq t$, otherwise (C2) implies that C is of size four with (u', v') an arc of C. Now if we consider the triangle $\{(v, u'), (u', v'), (v', v)\}$ with the cycle induced by the nodes u, u', v, w, we have the situation of case (c1), where v play the role of u and v' the role of w. Then from (ii) we must have $\delta^-(v) = \{(v', v)\}$, which is impossible since (u, v) is an arc with $u \neq v'$.

 * Condition (C1) implies that $t \in \dot{C}$. Denote by C' the cycle induced by $\{t, u, v, w\}$. This is a non-directed g-odd Y-cycle, then (C2) implies that C should be of size five, and $u' \in \hat{C}$. Then there is an arc (u', r). Condition (C2) implies $r \in C'$, but it is impossible because of (C1) and (iii). □

Lemma 24. *In case (b1) we have*

(i) $\delta^-(w) = \{(v, w)\}$,

(ii) $\delta^+(v) = \{(v, w)\}$,

(iii) $\delta^+(w) = \{(w, u)\}$.

Proof. (i) $\delta^-(w) = \{(v, w)\}$.
Consider an arc $(\bar{w}, w)$, $\bar{w} \neq v$. We need two observations.

– $\bar{w} \notin V(C) \setminus \{u, v, u', v'\}$.
To see this assume that $\bar{w} \in V(C) \setminus \{u, v, u', v'\}$. First notice that $\bar{w} \notin \hat{C}$, because H_1 would be present. Let P_1 be the path in C from $\bar{w}$ to v not containing u. Let P_2 be the path in C from $\bar{w}$ to u not containing v. Let C_1 be the cycle formed by P_1, (v, w) and $(\bar{w}, w)$. Let C_2 be the cycle formed by P_2, (w, u) and $(\bar{w}, w)$. Depending on the parity of P_1, one of these two cycles is a non-directed g-odd Y-cycle that leads to the violation of (C2). See Figure 22 in the Appendix.

– $\bar{w} = u'$. In this case the nodes $\{u', u, v, w\}$ induce a g-odd Y-cycle. Then condition (C2) implies that C is of size four and hence there is an arc in C between u' and v'.
  * $(u', v') \in A$. To satisfy conditions (C1) and (C2) we must have that G consist of the cycle C with the arcs $(u', w), (v, w), (w, u)$ and eventually the arc (w, v'), and arcs incident to u', (u', s) and (t, u') with $s, t \notin \{u, v, w, u', v'\}$. Suppose that the arc (w, v') exists. The unique odd directed cycles are $\Delta = \{(u, v), (v, w), (w, u)\}$ and $\Delta' = \{(v, w), (w, v'), (v', v)\}$. Moreover, the arcs $(u, v), (w, u), (v', v)$ and (w, v') are not tight, otherwise at least one of the inequalities associated with Δ and Δ' is violated. Therefore the labeling l in Figure 12 (a) produces a solution z_l that satisfies the same equation system satisfied by z this contradicts the fact that z is an extreme point.
  If the arc (w, v') does not exist, then the unique odd directed cycle is Δ and all its arcs are not tight. In this case we consider the labeling shown in Figure 12 (b).
  * $(v', u') \in A$. If we had the arc (w, v'), we would obtain H_3 after removing (u, v) and (u', u). Thus (w, v') is not present, and to satisfy (C1) and (C2), the only other possible arc is (u', s) with $s \notin \{u, v, w, u', v'\}$, see Figure 12 (c). The same labeling as before leads to a contradiction.

– $\bar{w} = v'$. This case reduces to the previous one. We rename u' by v', v' by u', v by w, w by u and u by v.

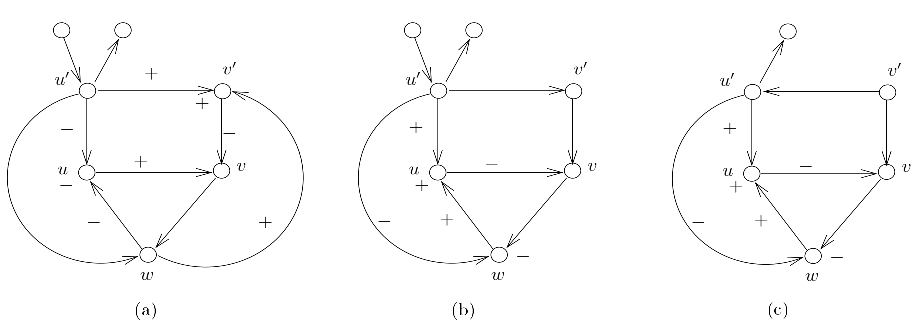

FIGURE 12. The different simple configurations of the graph G.

(ii) $\delta^+(v) = \{(v, w)\}$. We need several remarks.

– If there is an arc $(v, \bar{v})$, $\bar{v} \neq w$, condition (C1) implies $\bar{v} = u'$. This implies $u' \in \tilde{C}$, otherwise (C1) implies that $(u', v') \in A(C)$ and the cycle C with the arcs (v, u') and (v, w) induces the forbidden configuration H_3.
– If there is an arc $(u, \bar{u})$, then the cycle induced by $\{u', u, v, w\}$ with the arcs $(u, \bar{u})$ and (u, v)) induce H_3.
– If there is an arc $(w, \bar{w})$, condition (C1) implies that $\bar{w} = v'$. Then the cycle induced by $\{v', v, u, w\}$ with the arcs (v, u') and (v, w) induce H_3.
– Now assume that $(u', \bar{u}) \in A$, $\bar{u} \neq u$. From (C1) we have $\bar{u} = v'$. Then the cycle induced by $\{u', v', u, v\}$ with the arcs (v, u') and (v, w) induce H_3.
– Based on all remarks above, and (i), we apply Lemma 13, where the graph induced by $\{u, v, w, t\}$ in the lemma is exactly the graph induced by the nodes $\{u', w, v, u\}$. It is easy to see that the graphs G_1^* and G_2^* defined in Lemma 13, satisfy (C1) and (C2). From the induction hypothesis we have that both $PC_{p+1}(G_1^*)$ and $PC_p(G_2^*)$ are integral and hence Lemma 13 applies, but z is an extreme point of $PC_p(G)$ that contradicts this lemma.

(iii) $\delta^+(w) = \{(w, u)\}$. If there is an arc $(w, \bar{w})$, $\bar{w} \neq u$, we must have $\bar{w} = v'$, because of (C1). Then using (i) and (ii) Lemma 19 applies. □

Lemma 25. *In case (a1) we have*

(i) $\delta^-(u) = \{(w, u)\}$,
(ii) $\delta^+(w) = \{(w, u)\}$,
(iii) $\delta^-(w) = \{(v, w)\}$.
(iv) *The only common neighbor of u and v is w.*

Proof. (i) $\delta^-(u) = \{(w, u)\}$.

Assume that there is an arc $(\bar{u}, u)$, $\bar{u} \neq w$. Then (C1) implies $\bar{u} = v'$. The cycle induced by $\{u, v, v', w\}$ with the arcs (u, v) and (u, u') induce H_3.

(ii) $\delta^+(w) = \{(w, u)\}$.

If there is an arc $(w, \bar{w})$, with $\bar{w} \neq u$, then (C2) implies $\bar{w} \in V(C)$. This arc creates one or two new non-directed g-odd Y-cycles that together with (C2) lead to one of the following.
– either a violation of (C2),
– or a contradiction on the parity of C,
– or to the conclusion that $|V(C)| = 4$, see Figure 23 in the Appendix.

If $|V(C)| = 4$, then we must have (u', v') or (v', u') in $A(C)$. If (u', v') exists, to avoid H_2 we must have that $\bar{w} = u'$. Since C is a Y-cycle we must have an arc $(v', \bar{v})$. Since the cycle induced by $\{u, v, w, u'\}$ is a g-odd Y-cycle, (C2) implies that $\bar{v} = w$ but then H_1 is present. Now if (v', u') exists, we must have $\bar{w} = v'$ and again since C is a Y-cycle there is an arc $(u', \bar{u})$. We cannot have $\bar{u} = v$, otherwise H_1 is present, and since $\{u, v, w, v'\}$ induces a g-odd Y-cycle condition (C2) implies that $\bar{u} = w$. Now the cycle induces by $\{u, v, w, u'\}$ with the arcs (w, u) and (w, v') induce H_3.

(iii) $\delta^-(w) = \{(v, w)\}$.

If there is an arc $(\bar{w}, w)$, with $\bar{w} \neq v$, then (C1) implies $\bar{w} = u'$. Then because of (i) and (ii) we can apply Lemma 19.

(iv) Assume that s is a common neighbor of u and v, $s \neq w$. It follows from (i) that $(u, s) \in A$. We have two cases.
– If $(v, s) \in A$, we have an undirected g-odd Y-cycle C' induced by $\{u, v, w, s\}$. If $s \notin V(C)$, then there is at least one arc a of C in the path joining u' with v' that does not contain nor u nor v. This arc a together with C' would

violate (C2). If $s = v'$, then the cycle induce by $\{u, v, w, v'\}$ is a g-odd Y-cycle which implies that C is of size four and from (C1) we must have the arc (v', u') in C. So $u' \in \hat{C}$ and it must exists an arc $(u', \bar{u})$. From (iii) and (C2) we must have $\bar{u} = v$; but then H_1 is present. In the case $s = u'$, (C2) would lead to a contradiction. If $s \in V(C)$ and $s \neq u', v'$, since C' is a g-odd Y-cycle condition (C2) implies that both arcs (s, v') and (s, u') are in C. We have $v' \in \hat{C}$ and there is an arc (v', t) from (i) and (iii), $t \neq u, w$. Hence C' with the arc (v', t) violates (C2).

– If $(s, v) \in A$, then to avoid H_1 the arc (v', s) is in A. Then the arcs $(v', s), (u, s), (s, v)$ and (v, w) form H_1. □

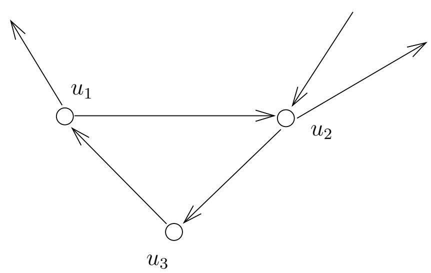

FIGURE 13

Lemma 26. *G does no contain a triangle $\Delta = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\}$, with*

(i) $\delta^+(u_3) = \delta^-(u_1) = \{(u_3, u_1)\}$,
(ii) $\delta^-(u_3) = \{(u_2, u_3)\}$,
(iii) *u_1 and u_2 have only u_3 as a common neighbor.*

See Figure 13.

Proof. Conditions (C1) and (C2) hold for $G(u_1, u_2)$ and $G(u_2, u_3)$. Both graphs satisfy the induction hypothesis, so $PC_{p+1}(G(u_1, u_2))$ and $PC_{p+1}(G(u_2, u_3))$ are integral.

Now we need the remarks below.

- Since Δ is the only directed cycle that contains (u_1, u_2), Lemma 16 and the fact that $PC_{p+1}(G(u_1, u_2))$ is integral imply

$$z(u_1, u_2) = z(u_2). \tag{22}$$

- Let $z' \in PC_{p+1}(G(u_1, u_2))$ be defined as $z'(u_1, u_2') = z(u_1, u_2)$, $z'(u_2') = 1$, $z'(u, v) = z(u, v)$, $z'(v) = z(v)$ for every other arc and node of $G(u_1, u_2)$. Since $PC_{p+1}(G(u_1, u_2)$ is integral, z' is a convex combination of $0 - 1$ vectors in $PC_{p+1}(G(u_1, u_2)$ that satisfy with equation every constraint that z' does.

  Among these vectors there is one, z^*, with $z^*(u_2, u_3) = 1$. This implies $z^*(u_3) = 1$, $z(u_3, u_1) = 0$. From this vector we define $\bar{z} \in PC_p(G)$, as follows:
  – if $z^*(u_1, u_2') = 0$, define $\bar{z}(u_1, u_2) = 0$, $\bar{z}(u_1) = z^*(u_1)$,
  – if $z^*(u_1, u_2') = 1$, define $\bar{z}(u_1, u_2) = 0$, $\bar{z}(u_1) = 1$,
  – $\bar{z}(u, v) = z^*(u, v)$, $\bar{z}(v) = z^*(v)$, for all other arcs and nodes of G.

  If $z(u_3, u_1) < z(u_1)$ then $\bar{z}$ satisfies with equation every constraint that z does, we have a contradiction, so we must assume that

$$z(u_3, u_1) = z(u_1). \tag{23}$$

- Since all arc variables have a nonzero value, and $z(\Delta) \le 1$, we should have

$$z(u_2, u_3) < z(u_3). \tag{24}$$

- Now we have to prove that $z(\Delta) = 1$. Notice that $|\delta^+(u_2)| \ge 2$, otherwise (22) and $z(\Delta) \le 1$ imply $z(u_3, u_1) = 0$. So assume that $(u_2, v) \in A$, $v \ne u_3$. Consider the vector $z' \in PC_{p+1}(u_1, u_2)$ defined above. Then z' is convex combination of $0-1$ vectors in $PC_{p+1}(u_1, u_2)$ that satisfy with equation every constraint that z' does. Among them there is one, z^* say, with $z^*(u_2, v) = 1$. This implies $z^*(u_2) = z^*(u_2, u_3) = 0$. From z^* we define $\bar{z}$ as follows.
  - If $z^*(u_3) = 1$, $z^*(u_3, u_1) = z^*(u_1) = 0$, we have two cases:
    * If $z^*(u_1, u_2') = 1$, then set $\bar{z}(u_1, u_2) = 0$, $\bar{z}(u_1) = \bar{z}(u_3, u_1) = 1$, $\bar{z}(u_3) = 0$; $\bar{z}(u, v) = z^*(u, v)$, $\bar{z}(v) = z^*(v)$, for all other arcs and nodes of G. Notice that in this case any odd directed cycle containing (u_2, u_3) and (u_3, u_1) and different from Δ is not tight for z.
    * If $z^*(u_1, u_2') = 0$, then set $\bar{z}(u_1, u_2) = 0$, and $\bar{z}(u, v) = z^*(u, v)$, $\bar{z}(v) = z^*(v)$, for all other arcs and nodes of G.
  - If $z^*(u_3) = 0$, $z^*(u_3, u_1) = z^*(u_1) = 1$, $z^*(u_1, u_2') = 0$, then set $\bar{z}(u_1, u_2) = 0$ and $\bar{z}(u, v) = z^*(u, v)$, $\bar{z}(v) = z^*(v)$, for all other arcs and nodes of G.

  It is easy to see that $\bar{z} \in PC_p(G)$. If $z(\Delta) < 1$ then each inequality that is tight for z is also tight for $\bar{z}$, and we have a contradiction. So we have to assume that

$$z(\Delta) = 1. \tag{25}$$

- If (u_2, u_3) does not belong to an odd directed cycle other than Δ, then (24) and Lemma 16 contradict the fact that $PC_{p+1}(G(u_2, u_3))$ is integral. So we assume that (u_2, u_3) belongs to an odd directed cycle C', different from Δ. Furthermore

$$z(A(C')) = \frac{|C'| - 1}{2}. \tag{26}$$

  Now we discuss some properties of C'.
  - Let P' be the directed path in C' from u_1 to u_2, not containing u_3. Then P' together with the arc (u_1, u_2) form a non-directed g-odd Y-cycle C''. Condition (C2) implies that every arc in A has at least one endnode in $V(C'')$.
  - Let $(s, t) \in A$, with $t \in V(C'')$, then in order to avoid H_1, we should have $s \in V(C'')$.
  - For the arc (s, t) mentioned above, let (t, u), (u, s) be the two arcs in the path from t to s in C'. If we remove (t, u) and (u, s) from C' and add (s, t) we obtain a non-directed g-odd Y-cycle D that intersects the triangle $\Delta' = \{(s, t), (t, u), (u, s)\}$ as in case (c1), and Lemma 23 applies.
  - This implies that any other directed cycle is a triangle, and it intersects C'' as in (c1). Therefore Lemma 23 applies.
- Now consider a graph G' obtained by shrinking Δ. That is we remove the nodes u_1, u_2, u_3 and add a node $\bar{u}$ that is incident to the arcs in G that were incident to u_1 and u_2, except the arc (u_1, u_2). Because of hypothesis (iii), we obtain an oriented graph with no multiple arcs. Since $\delta^-(u_1) = \{(u_3, u_1)\}$, condition (C1) is satisfied. It is also easy to see that condition (C2) remains satisfied. Let z' be the restriction of z to G' and $z'(\bar{u}) = z(u_2)$.

Now we prove that $z'(\bar{u}) + z'(\delta^+(\bar{u})) = 1$ and $\sum z'(u) = p - 1$. We have

$$z(u_3) + z(u_3, u_1) = 1 \tag{27}$$

$$z(u_1) + z(u_1, u_2) + z(\delta^+(u_1) \setminus \{(u_1, u_2)\}) = 1 \tag{28}$$

$$z(u_2) + z(u_2, u_3) + z(\delta^+(u_2) \setminus \{(u_2, u_3)\}) = 1. \tag{29}$$

To see that $\sum z'(u) = p - 1$ we combine (27) and (23).

We have

$$z'(\bar{u}) + z'(\delta^+(\bar{u})) = z(u_2) + z(\delta^+(u_1) \setminus \{(u_1, u_2)\}) + z(\delta^+(u_2) \setminus \{(u_2, u_3)\}),$$

then the combination of (28), (29), (23) and (25) gives $z'(\bar{u}) + z'(\delta^+(\bar{u})) = 1$.

- In the graph G' the cycle C' is transformed into an odd directed cycle $\bar{C}$, where $A(\bar{C}) = A(C') \setminus \{(u_2, u_3), (u_3, u_1)\}$. Since $z(u_2, u_3) + z(u_3, u_1) < 1$, then (26) implies

$$z'(A(\bar{C})) > \frac{|\bar{C}| - 1}{2}. \tag{30}$$

We conclude that $z' \notin PC_{p-1}(G')$ and $z' \in P_{p-1}(G')$. Since all triangles satisfy the Lemma hypothesis, we can repeat this procedure k times, where k is the number of triangles. Then we obtain a graph G^* with no triangles, and and a vector z^* with $z^* \notin PC_{p-k}(G^*)$ and $z^* \in P_{p-k}(G^*)$. The graph G^* does not contain an odd directed cycle other than $\bar{C}$. If condition (ii) of Theorem 9 was violated by $\bar{C}$ and some arc, then condition (C2) would be violated by this same arc and the cycle C'' defined above. Thus we have a contradiction, because it follows from Theorem 9 that $P_{p-k}(G^*)$ is integral and therefore $PC_{p-k}(G^*) = P_{p-k}(G^*)$. □

3.3. **Final part in the proof of Theorem 5.** If there is a triangle, condition (C2) implies that it should intersect C. Now we consider the different intersection cases.

- In Case (a1) we use Lemmas 25 and 26.
- In Case (b1) Lemmas 20 and 24 give a contradiction.
- In Case (c1) Lemmas 23 and 26 give a contradiction.
- In Case (d1), to avoid H_1 we should have $u' = v'$, but in this case C has size three.
- In Case (d2), to avoid H_1 we should have $u' = v'$. This implies that C is not a Y-cycle, otherwise H_3 is present.
- In Case (c2) consider the cycle C' obtained from C by removing (v, w), (w, u) and adding (u, v). Then C' is also a g-odd Y-cycle and is treated as in Case (c1).

Now it remains the Cases (a2) and (b2).

3.3.1. *Treatment of Case (a2).* We need several remarks.

(1) We should have $\delta^-(u) = \{(w, u)\}$. Otherwise if there is an arc $(\bar{u}, u)$, we should have $\bar{u} = v'$, to avoid H_1. But then H_3 is present.
(2) Also we have $\delta^-(w) = \{(v, w)\}$. Otherwise consider an arc $(\bar{w}, w)$. To avoid H_1 we should have $u' = \bar{w}$. It follows from (1) and Lemma 19 that there is an arc $(w, \bar{w})$, with $\bar{w} \neq u$. But then H_3 is present.
(3) Now we study $\delta^-(v)$.
    - Assume that $\delta^-(v) = \{(u, v)\}$. From (1) and (2) we have $\delta^-(\{u, v, w\}) = \emptyset$. To apply Lemma 22 assume first that two nodes in $\{u, v, w\}$ have a common neighbor $p \notin \{u, v, w\}$. Since p is not pendent, there is an arc

(p,q), $q \notin \{u,v,w\}$. Thus $\{p,u,v,w\}$ induces an undirected g-odd Y-cycle. If $p \notin \{u',v'\}$, condition (C2) implies that p is a neighbor of u' and v' in C. And condition (C1) implies that $p \in \dot{C}$. This implies that C is g-even. Thus we should have $p \in \{u',v'\}$. In this case since $\delta^-(\{u,v,w\}) = \emptyset$, H_1 must be present.

Thus Lemma 22 implies that at least one of the arcs in the triangle Δ, induced by $\{u,v,w\}$ is not tight. Denote by (s,t) this arc. Let us see that $G(s,t)$ satisfies the induction hypothesis. Obviously, it satisfies (C1) and has less triangles than G. Now we need three remarks to discuss (C2).

- If t has no neighbor outside the triangle then $t = w$, and (C2) follows easily.
- If there is a node r in Δ with no neighbor in $V \setminus \{u,v,w\}$ then $r = w$, and the arc (v,w) must be non tight, otherwise the inequality associated with Δ is violated. Then the remark above applies.
- It remains the case when the three nodes in Δ have a neighbor in $V \setminus \{u,v,w\}$. Then Lemma 21 implies that (C2) is satisfied by $G(s,t)$.

Thus the induction hypothesis implies that $PC_{p+1}(G(s,t))$ is integral. Since (s,t) does not belong to another directed cycle, Lemma 16 implies that $PC_{p+1}(G(s,t))$ is not integral, a contradiction.

– Assume now that $\delta^-(v) \neq \{(u,v)\}$. So there is an arc $(\bar{v},v)$, $\bar{v} \neq u$. In order to avoid H_1 and because of (1), there is a unique arc leaving v', this is $(v',\bar{v})$. If $\bar{v}$ is not in C, then the cycle C and the triangle induced by $\{v',\bar{v},v\}$ correspond to the case (c1) already treated. Thus now we assume that $\bar{v} \in C$. We have to study $\delta^+(w)$.
  - Assume that $\delta^+(w) = \{(w,u)\}$. To apply Lemma 26, first we have to show that the unique common neighbor of u and v is w. Suppose that s is another common neighbor. From (1) we have that $(u,s) \in A$. Here we have two cases:
    - If $(s,v) \in A$, to avoid H_2 we should have $\bar{v} = s$. Then the graph H_1 is given by $(v',\bar{v}), (u,\bar{v}), (\bar{v},v)$ and (v,w).
    - If $(v,s) \in A$ then the nodes $\{u,v,w,s\}$ induces a g-odd Y-cycle C'.
      If s is not in C then $(v',\bar{v})$ and C' violate (C2).
      If s is in C, then we have two more cases. If $s \neq v'$ then $(v',\bar{v})$ and C' violate (C2). If $s = v'$, then (C2) implies $\bar{v} = u$ and C is g-even.

    We have shown that the unique common neighbor of u and v is w, then Lemma 26 gives a contradiction.
  - Now we assume that $\delta^+(w) \neq \{(w,u)\}$. Then $(w,\bar{w}) \in A$, with $\bar{w} \neq u$. To avoid H_1 we should have $\bar{w} = \bar{v}$. Then the arcs $(v,v'), (v',\bar{v}), (v,w)$ and $(w,\bar{v})$ form a g-odd cycle C'. Since there are already two arcs entering $\bar{v}$, and in order to avoid H_2 there are no more arcs entering $\bar{v}$. Since $\bar{v}$ is in C, there is an arc, other than $(\bar{v},v)$, leaving $\bar{v}$. Thus C' is a Y-cycle, and since it is g-odd the arc (u,u') violates (C2).

3.3.2. *Treatment of Case (b2).* We need several remarks.

(1) We should assume that $\delta^+(w) = \{(w,u)\}$.
Assume the opposite, i.e., $(w,\bar{w}) \in A$, with $\bar{w} \neq u$. In order to avoid H_1, we need $\bar{w} = v'$. If $v' \in \dot{C}$ then the cycle C together with the triangle induced by

$\{v', v, w\}$ correspond to the Case (a2) already treated. If $v' \in \tilde{C}$ we have the Case (c2).

(2) We should assume that $\delta^+(v) = \{(v, w)\}$.

If there is an arc $(v, \bar{v})$, to avoid H_1 we should have $\bar{v} = u'$. If $u' \in \dot{C}$ then to avoid H_1 we should have that $(u', v') \in C$, but then C would be g-even. If $u' \in \tilde{C}$, then the path in C between u' and v containing v', and the arc (v, u') form a g-odd Y-cycle C'. The cycle C' and the arc (w, u) violate (C2).

(3) If $\delta^-(w) = \{(v, w)\}$, then Lemma 20 gives a contradiction.

(4) Assume that $\delta^-(w) \neq \{(v, w)\}$. So there is an arc $(\bar{w}, w)$ with $\bar{w} \neq v$. Below we give some observations about this arc.

- If $\bar{w} \in \{u', v'\}$ there is a g-odd Y-cycle C' that together with condition (C2) forces C to be g-even.
- If $\bar{w}$ is adjacent to either u' or v', there is g-odd Y-cycle C' that intersects the triangle Δ as in (b1).
- If $\bar{w} \in V(C) \setminus \{u, v, u', v'\}$, we should have $\bar{w} \notin \hat{C}$, otherwise H_1 is present. This creates a new non-directed g-odd Y-cycle that leads to a violation of (C2) or to a contradiction on the parity of C. See Figure 24 in the Appendix.
- If $\bar{w}$ is adjacent to a node in $V(C) \setminus \{u, v, u', v'\}$, we should have $\bar{w} \notin \hat{C}$, otherwise (C1) is violated. This creates a new non-directed g-odd Y-cycle that leads to a violation of (C2), see Figures 25 and 26 in the Appendix; or we could have one of the configurations in Figure 14.

  To treat these four cases we need the following remarks.

  * There is no arc $(u, \bar{u})$ with $\bar{u} \neq v$. Otherwise, the only possibility to avoid H_1 is $\bar{u} = \bar{w}$, but then the cycle C with the triangle induced by $\{u, \bar{w}, w\}$ correspond to the case (b1) treated before. With this remark and because of (1) and (2), we can assume that the arcs in the triangle do not belong to any other directed cycle and none of them is tight.
  * Let $R = \{r, s, t\}$ be the set of nodes in C not in Δ. If there is any other directed cycle D, it follows from (C2) that every arc in D should be incident to at least one node in R. Thus D has size at most five. If D has size five, one arc in C creates a triangle with two arcs of D that intersects C as in one of the cases already treated. If D is a triangle, it also intersects C as in one of the cases already treated.
  * Thus we can assume that the only directed odd cycle is the triangle Δ. Also condition (C1) implies that there is no other arc entering a node that has a non-zero label in the figure. Thus the labels in the figure give a new vector that satisfies with equation the same constraints as z, and we have a contradiction.
- We cannot have the arcs $(u, \bar{w})$, $(\bar{w}, u)$, $(\bar{w}, v)$ and $(v, \bar{w})$. From the discussion above we may assume that $\bar{w} \notin V(C)$. Obviously the arcs $(\bar{w}, u)$ and $(\bar{w}, v)$ cannot exist otherwise H_2 is present. If $(u, \bar{w})$ exists we would have a triangle that intersects C as in (b1). If $(v, \bar{w})$ exists, the graph H_1 would be present.

Thus we have that $\bar{w}$ is not in $V(C)$ and is only adjacent to w. To avoid H_1 we should have $\delta^+(u) = \{(u, v)\}$.

Condition (C2) implies that any non-directed g-odd Y-cycle should contain w, and therefore it should contain (u', u) and (v', v). Thus the graphs $G(u', u)$ and $G(v', v)$ do not contain a non-directed g-odd Y-cycle, and satisfy (C1). Theorem 3 implies that $PC_{p+1}(G(u', u))$ and $PC_{p+1}(G(v', v))$ are integral. Moreover, the arcs (u', u) and (v', v) do not belong to any odd directed cycle, then Lemma 16 implies that $z(u',u) = z(u)$ and $z(v',v) = z(v)$. These are the only inequalities containing $z(u',u)$ and $z(v',v)$ in the definition of $PC_p(G)$ that are satisfied with equation by z. Lemma 14 implies that $z(u',u) = z(u) = 1/2$ and $z(v',v) = z(v) = 1/2$. Thus $z(u,v) = z(v,w) = 1/2$, and since $z(\Delta) \le 1$ we have $z(w,u) = 0$, a contradiction.

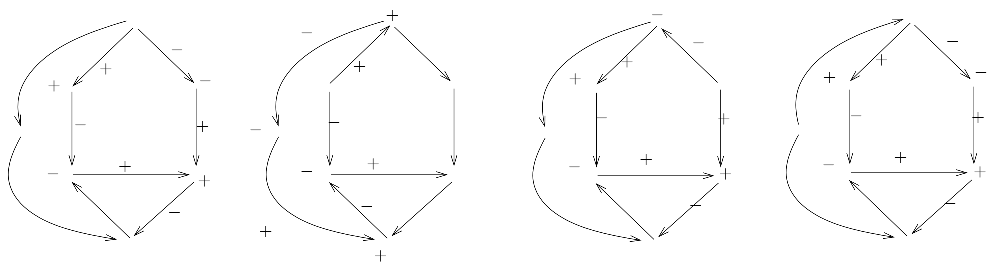

Figure 14

4. Concluding remarks

We have completed the characterization of the oriented graphs for which $PC_p(G)$ defines the p-median polytope. This was done in two parts: In [2] we treated graphs without a non-directed g-odd Y-cycle, and graphs with no triangle. In this second paper we treated graphs with a non-directed g-odd Y-cycle and with triangles. Here we had to use induction on the number of triangles.

Odd cycle inequalities are among the simplest classes of inequalities that one can use to improve a linear relaxation of the p-median problem. Their separation problem can be easily reduced to a shortest path problem. Here we identified the oriented graphs for which these are the only extra inequalities needed.

References

[1] M. Baïou and F. Barahona, "On the linear relaxation of the p-median problem", *Discrete Optimization* 8 (2011) 344-375.

[2] M. Baïou and F. Barahona, "On the p-median polytope and the directed cycle inequalities I: triangle-free oriented graphs", IBM Research Report 2013.

5. Appendix

Lemma 27. *Consider a triangle $\Delta = \{(u,v),(v,w),(w,u)\}$, and a non-directed g-odd Y-cycle C. If none of the arcs of Δ is in C, but its three nodes are in C, then either the graph H_1 is present, or there is another non-directed g-odd Y-cycle containing one arc of Δ.*

Proof. If $w \in \tilde{C}$ either the graph H_1 is present, or there is another non-directed g-odd Y-cycle containing one arc of Δ. See Figure 15.

If $w \in \dot{C}$, either H_1 is present or there is another non-directed g-odd Y-cycle of smaller size, containing one arc of Δ. See Figure 16.

□

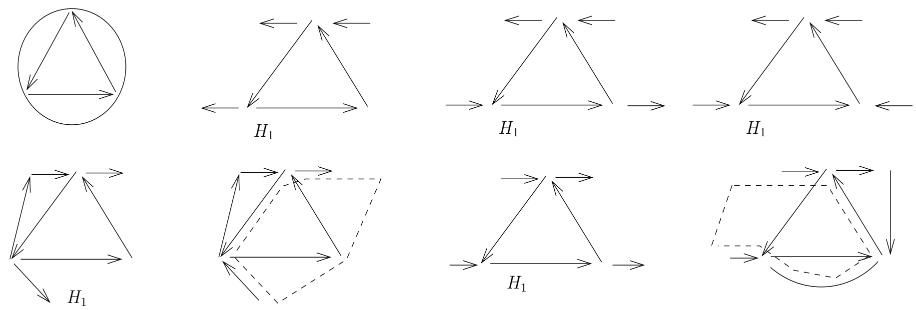

FIGURE 15

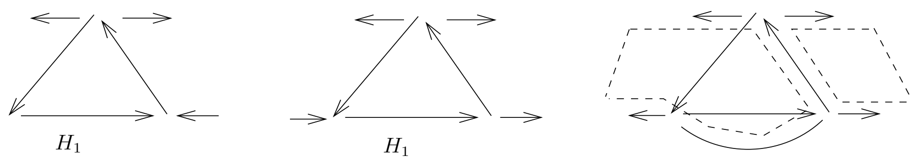

FIGURE 16

Lemma 28. *Consider a triangle* $\Delta = \{(u,v),(v,w),(w,u)\}$*, and a non-directed g-odd* Y*-cycle* C*. If none of the arcs of* Δ *is in* C*, and the nodes* u *and* w *are in* C*, then either the graph* H_1 *is present, or there is another non-directed g-odd* Y*-cycle containing one or two arcs of* Δ*.*

Proof. If $w \in \tilde{C}$ either the graph H_1 is present, or there is another non-directed g-odd Y-cycle containing one arc of Δ. See Figure 17.

If $w \in \dot{C}$, then there is another non-directed g-odd Y-cycle containing two arcs of Δ. See Figure 18.

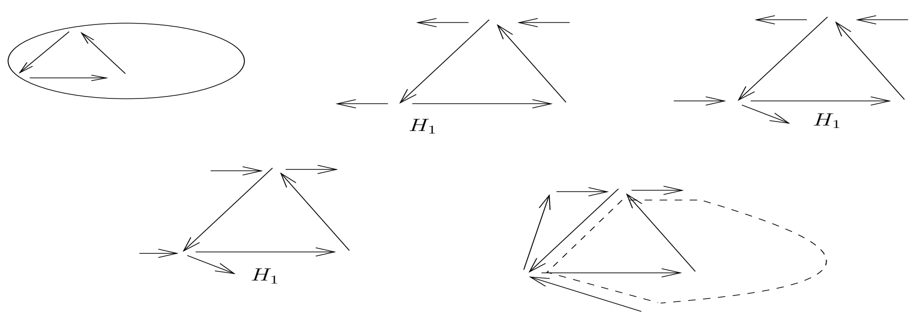

FIGURE 17

Lemma 29. *Consider a triangle* $\Delta = \{(u,v),(v,w),(w,u)\}$*. If* Δ *intersects a non-directed g-odd* Y*-cycle* C *in the arc* (u,v)*, and the node* w *is in* C*, then either condition (C1) is violated, or there is another non-directed g-odd* Y*-cycle that either intersects* Δ *in two arcs, or in one arc and no other node.*

Proof. The proof consists of two parts.

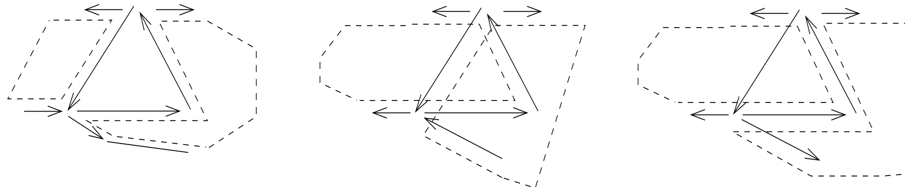

FIGURE 18

- Suppose that $w \in \tilde{C}$. Then either the graph H_1 or H_3 is present, or there is a non-directed g-odd Y-cycle that intersects Δ in one arc and no other node. In Figures 19 and 20 we show the possible configurations.

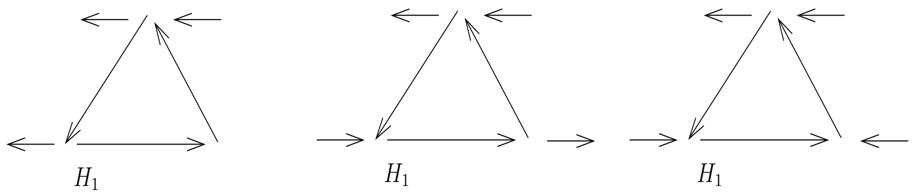

FIGURE 19

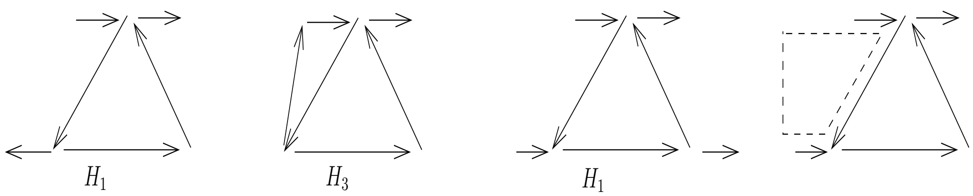

FIGURE 20

- Suppose that $w \in \dot{C}$. Then either we have H_1 or there is another non-directed g-odd Y-cycle that either intersects Δ in two arcs, or in one arc and no other node. In Figure 21 we show the possible configurations.

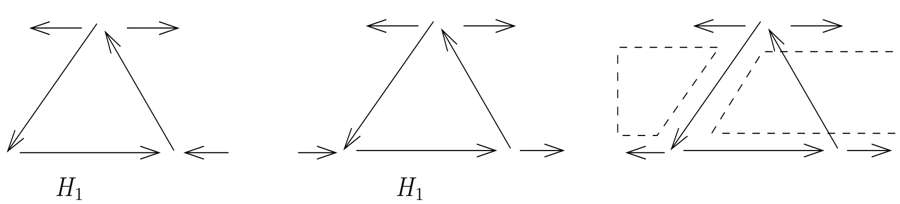

FIGURE 21

□

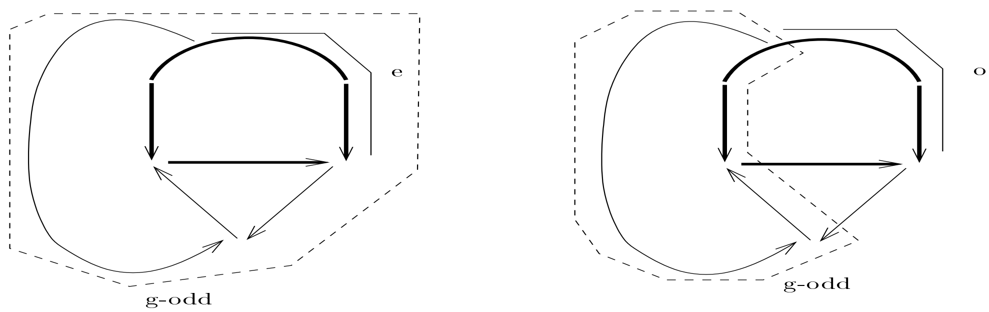

FIGURE 22. Case (b1). The labels "o" or "e" refer to the parity of a path.

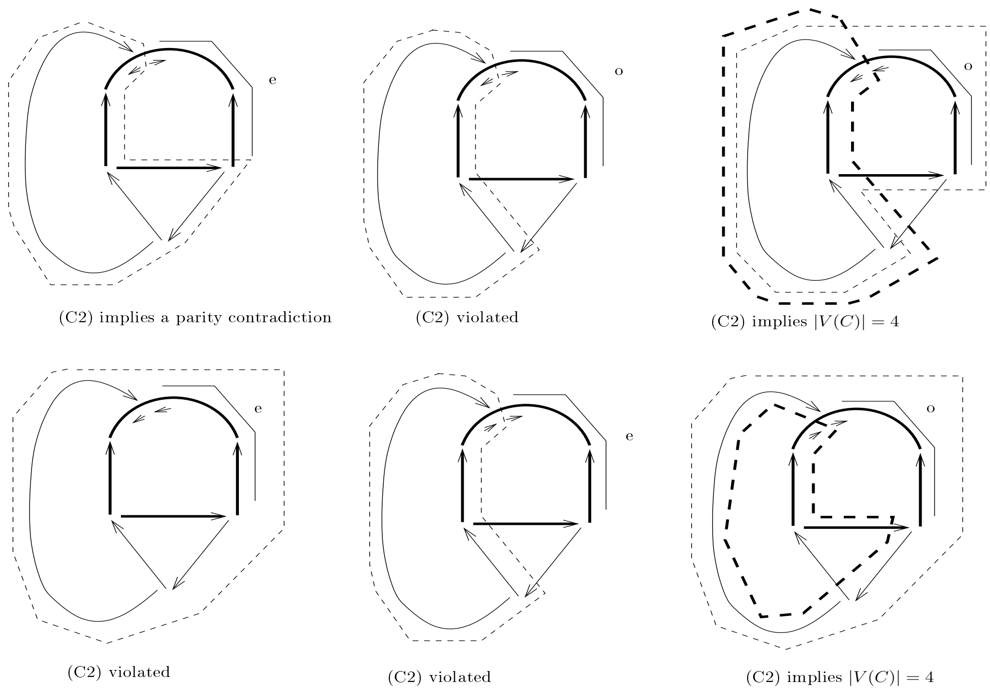

FIGURE 23. Case (a1). The labels "o" or "e" refer to the parity of a path. Non-directed g-odd Y-cycles are depicted with dashed lines.

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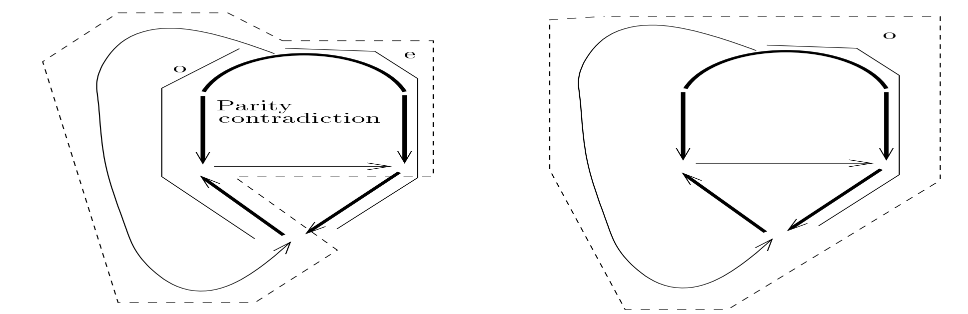

FIGURE 24. Case (b2). The labels "o" or "e" refer to the parity of a path.

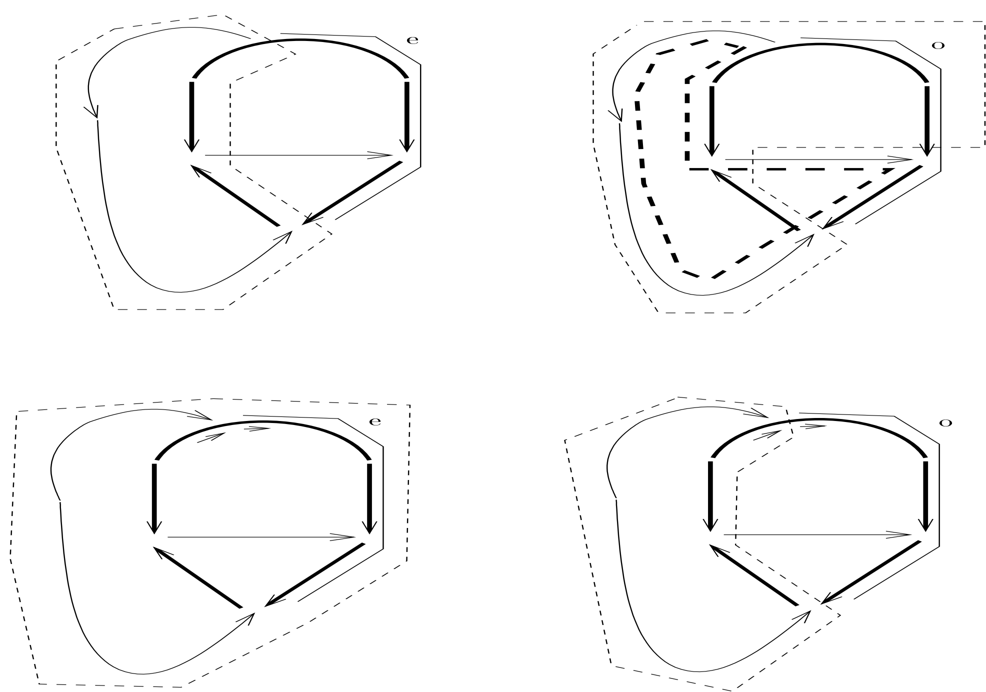

FIGURE 25. Case (b2). The labels "o" or "e" refer to the parity of a path. Non-directed g-odd Y-cycles are depicted with dashed lines.

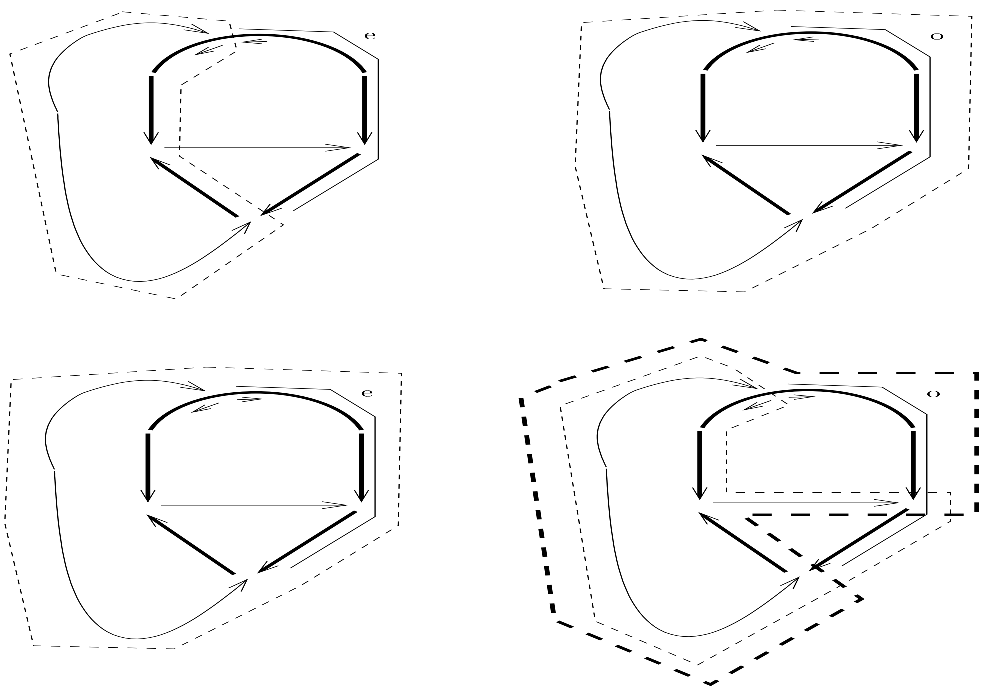

FIGURE 26. Case (b2). The labels "o" or "e" refer to the parity of a path. Non-directed g-odd Y-cycles are depicted with dashed lines.