RC25430 (WAT1311-165) November 20, 2013
Mathematics

IBM Research Report

On Complexity of LP&SDP-Based Stability Certificates for Switched Linear Systems

Amir Ali Ahmadi
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 208
Yorktown Heights, NY 10598
USA

Raphaël M. Jungers
Université catholique de Louvain
Louvain-la-Neuve, Belgium

On Complexity of LP&SDP-Based Stability Certificates for Switched Linear Systems ⋆

Amir Ali Ahmadi * **Raphaël M. Jungers** **

* *Department of Business Analytics and Mathematical Sciences of the IBM Watson Research Center. Email:* `aaa@us.ibm.com`*.*
** *ICTEAM Institute, Université catholique de Louvain, Louvain-la-Neuve, Belgium. Email:* `raphael.jungers@uclouvain.be`

Abstract: We show that for any positive integer d, there are families of switched linear systems—in fixed dimension and defined by two matrices only—that are stable under arbitrary switching but do not admit (i) a polynomial Lyapunov function of degree $\leq d$, or (ii) a polytopic Lyapunov function with $\leq d$ facets, or (iii) a piecewise quadratic Lyapunov function with $\leq d$ pieces. This implies that there cannot be an upper bound on the size of the linear and semidefinite programs that search for such stability certificates. Several constructive and non-constructive arguments are presented which connect our problem to known (and rather classical) results in the literature regarding the finiteness conjecture, undecidability, and non-algebraicity of the joint spectral radius. In particular, we show that existence of a sum of squares Lyapunov function implies the finiteness property of the optimal product.
Index terms: stability of switched systems, linear difference inclusions, the finiteness conjecture of the joint spectral radius, convex optimization for Lyapunov analysis.

1. INTRODUCTION

We are concerned in this paper with one of the most basic and simple to describe classes of hybrid dynamical systems, namely those that undergo arbitrary switching between a finite set of discrete time linear dynamical systems. In this setting, the input to our problem is a set of m real $n \times n$ matrices $\Sigma := \{A_1, \ldots, A_m\}$. This set describes a switched linear system of the form

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{1}$$

where k is the index of time and $\sigma : \mathbb{Z} \to \{1, ..., m\}$ is a map from the set of integers to the set $\{1, \ldots, m\}$. A basic notion of stability is that of *absolutely asymptotically stable* (AAS), also referred to *asymptotic stability under arbitrary switching* (ASUAS), which asks whether all initial conditions in $\mathbb{R}^n$ converge to the origin for *all* possible switching sequences. It is not difficult to show that absolute asymptotic stability of (1) is equivalent to absolute asymptotic stability of the linear difference inclusion

$$x_{k+1} \in \text{co}\Sigma \; x_k, \tag{2}$$

where coΣ here denotes the convex hull of the set Σ. Among other motivations, dynamical systems in (1) or (2) model a linear system which is subject to time-dependent uncertainty. See for instance Liberzon [2003], Shorten et al. [2007], or Jungers [2009] for more applications in systems and control.

When the set Σ consists of a single matrix A (i.e., $m = 1$), we are of course in the simple case of a linear system where asymptotic stability is equivalent to the spectral radius of A having modulus less than one. This condition is also equivalent to existence of a *quadratic* Lyapunov function. When $m \geq 2$, however, no efficiently checkable criterion is known for AAS. Arguably, the most promising approaches in the literature have been to use convex optimization (typically linear programming (LP) or semidefinite programming (SDP)) to construct Lyapunov functions that serve as certificates of stability. The most basic example is that of a *common quadratic Lyapunov function* (CQLF), which is a positive definite quadratic form $x^T Q x$ that decreases with respect to all m matrices, i.e., satisfies $x^T(A_i^T Q A_i - Q)x < 0, \forall x \in \mathbb{R}^n, i = 1, \ldots, m$. On the positive side, the search for such a quadratic function is efficient numerically as it readily provides a semidefinite program. On the negative side, and in contrast to the case of linear systems, existence of a CQLF is a sufficient but *not* necessary condition for stability. Indeed, a number of authors have constructed examples of AAS switched systems which do not admit a CQLF and studied various criteria for existence of a CQLF (Ando and Shih [1998], Dayawansa and Martin [1999], Mason and Shorten [2004], Olshevsky and Tsitsiklis [2008]).

To remedy this shortcoming, several richer and more complex classes of Lyapunov functions have been introduced. We list here the five that are perhaps the most ubiquitous:

Polynomial Lyapunov functions. A homogeneous [1] multivariate polynomial $p(x)$ of some even degree d is a *polynomial Lyapunov function* for (1) if it is positive

⋆ A.A.A. is a Goldstine Fellow and with the Mathematical Programming group of IBM Research. R.J. is supported by the Communauté francaise de Belgique - Actions de Recherche Concertées, and by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. R.J. is a F.R.S.-FNRS Research Associate.

[1] Since the dynamics in (1) is homogeneous, there is no loss of generality in parameterizing our Lyapunov functions as homogeneous functions. Also, we drop the prefix "common" from the terminology "common polynomial Lyapunov function" as it is implicit that our Lyapunov functions are always common to all m matrices A_i in Σ.

definite [2] and makes $p(x) - p(A_i x)$ positive definite for $i = 1, \dots, m$.

Although this is a rich class of functions, a numerical search for polynomial Lyapunov functions is an intractable task even when the degree d is fixed to 4. In fact, even testing if a given quartic form is positive definite is NP-hard in the strong sense (see, e.g.,Ahmadi [2012]). A popular and more tractable subclass of polynomial Lyapunov functions is that of sum of squares Lyapunov functions.

Sum of squares (sos) Lyapunov functions. A homogeneous polynomial of some even degree d is a *sum of squares (sos) Lyapunov function* for (1) if p is positive definite and a sum of squares [3], and if all the m polynomials $p(x) - p(A_i x)$, $i = 1, \dots, m$ are also positive definite and sums of squares.

For any fixed degree d, the search for an sos Lyapunov function of degree d is a semidefinite program of size polynomial in the input. When $d = 2$, this class coincides with CQLFs as nonnegative quadratic forms are always sums of squares. Moreover, existence of an sos Lyapunov function is not only sufficient but also necessary for AAS of (1) (Parrilo and Jadbabaie [2008]). This of course implies that existence of polynomial Lyapunov functions defined above is also necessary and sufficient for stability.

Polytopic Lyapunov functions. A polytopic Lyapunov function V for (1) with d pieces is one that is a pointwise maximum of d linear functions:

$$V(x) := \max_{i=1\dots,d} |c_i^T x|,$$

where $c_1, \dots, c_m$ span $\mathbb{R}^n$. The sublevel sets of such functions are polytopes, justifying their name. Polytopic Lyapunov functions (with enough number of pieces) are also necessary and sufficient for absolute asymptotic stability. One can use linear programming to search for subclasses of these Lyapunov functions. These subclasses are big enough to also comprise a necessary and sufficient condition for stability (see Lin and Antsaklis [2005], Polański [1997, 2000]).

Max-of-quadratics Lyapunov functions. A *max-of-quadratics* Lyapunov function V for (1) with d pieces is one that is a pointwise maximum of d positive definite quadratics:

$$V(x) := \max_{i=1,\dots,d} x^T Q_i x,$$

where $Q_i \succ 0$. The sublevel sets of such functions are intersections of ellipsoids.

Min-of-quadratics Lyapunov functions. A *min-of-quadratics* Lyapunov function V for (1) with d pieces is one that is a pointwise minimum of d positive definite quadratics:

$$V(x) := \min_{i=1,\dots,d} x^T Q_i x,$$

where $Q_i \succ 0$. The sublevel sets of such functions are unions of ellipsoids.

By a **piecewise quadratic Lyapunov function**, we mean one that is either a max-of-quadratics or a min-of-quadratics. Both of these families are known to provide necessary and sufficient conditions for AAS. Several references in the literature produce semidefinite programs that can search over a subclass of max-of-quadratics or min-of-quadratics Lyapunov functions (see Goebel et al. [2006]). These subclasses alone also provide necessary and sufficient conditions for AAS. A unified framework to produce such SDPs is presented in Ahmadi et al. [2013], where a recipe for writing down stability proving linear matrix inequalities is presented based on some connections to automata theory.

For all classes of functions we presented, one can think of d as a *complexity parameter* of the Lyapunov functions. The larger the parameter d, the more complex our Lyapunov function would look like and the bigger the size of an LP or an SDP searching for it would need to be.

1.1 Motivation and contributions

Despite the encouraging fact that all five classes of Lyapunov functions mentioned above provide necessary and sufficient conditions for AAS of (1) that are amenable to computational search via LP or SDP, all methods offer an *infinite hierarchy* of algorithms, for increasing values of d, leaving unclear the natural questions: How high should one go in the hierarchy to obtain a proof of stability? How does this number depend on n (the dimension) and m (the number of matrices)? Unlike the case of CQLF which is ruled out as a necessary condition for stability through several counterexamples in the literature, we are not aware of that many counterexamples that rule out more complicated Lyapunov functions. For example, is there an example of a set of matrices that is AAS but does not admit a polynomial Lyapunov function of degree 4, or 6, or 200? [4] Or, is there an example of a set of matrices that is AAS but does not admit a piecewise quadratic Lyapunov function with 200 pieces? If such sets of matrices exist, how complicated do they look like? How many matrices should they have and in what dimensions should they appear?

In this paper we give an answer to these questions, providing constructive and non-constructive arguments for existence of "families of vary bad matrices", i.e., those forcing the complexity parameter d of all Lyapunov functions to be arbitrarily large, even for fixed n and m (in fact, even for the minimal situation $n = m = 2$). The formal statement is given in Theorem 1 below.

It is important to remark that the families of matrices we present have already appeared in rather well-established literature, though for different purposes. These matrices have to do with the "non-algebraicity" and the "finiteness property" of the notion of *joint spectral radius* (JSR) (see Sections 2 and 3 for definitions). This leaves us with the much simplified task of establishing a formal connection between these two concepts and that of complexity of Lyapunov functions. We hope that clarifying these connections sheds new light on the intrinsic relationship between

[2] A form (i.e., homogeneous polynomial) p is *positive definite* if $p(x) > 0$ for all $x \neq 0$.

[3] A polynomial p is a sum of squares if it can be written as $p = \sum_i q_i^2$ for some polynomials q_i.

[4] The largest degree existing counterexample that we know of is one of our own, appearing in Ahmadi and Jungers [2013], which is a pair of AAS 2×2 matrices with no polynomial Lyapunov function of degree 14.

the JSR and the stability question for switched linear systems. Indeed, many of the results that we refer to in the literature on the JSR appear much before counterexamples to existence of CQLF in the switched system literature.

Theorem 1. *For any positive integer d, the following families of matrices (parameterized by k) include switched systems that are asymptotically stable under arbitrary switching but do not admit (i) a polynomial (hence SOS) Lyapunov function of degree $\leq d$, or (ii) a polytopic Lyapunov function with $\leq d$ facets, or (iii) a max-of-quadratics Lyapunov function with $\leq d$ pieces, or (iv) a min-of-quadratics Lyapunov function with $\leq d$ pieces:*

(1) $(1-\frac{1}{k})\{A_1, A_2\}$, *with*

$$A_1 = \frac{(1-t^4)}{(1-3\pi t^3/2)}\begin{bmatrix}\sqrt{1-t^2} & -t\\ 0 & 0\end{bmatrix},$$

$$A_2 = (1-t^4)\begin{bmatrix}\sqrt{1-t^2} & -t\\ t & \sqrt{1-t^2}\end{bmatrix},$$

where $t = \sin\frac{2\pi}{2k+1}$ *and* $k = 1, 2, \ldots$.

(This family appears in the work of Kozyakin [1990] as an example demonstrating that the joint spectral radius is not a semialgebraic quantity; see Section 2.)

(2) $(1-\frac{1}{k})\{A_1, \ldots, A_m\}$, $k = 1, 2, \ldots$, *where* $A_1, \ldots, A_m$ *are any fixed set of matrices with JSR equal to* 1 *that provide a counterexample to the finiteness conjecture (see Blondel et al. [2003], Bousch and Mairesse [2002]); for example, those in Hare et al. [2011]:*

$$A_1 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, A_2 = \alpha_*\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},$$

where

$\alpha_* \simeq 0.749326546330367557943961948091344672091...$

(3) $(1-\frac{1}{k})\{A_1, A_2\}$, *with*

$$A_1 = \alpha^k\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ A_2 = \alpha^{-1}\begin{bmatrix}\cos\dfrac{\pi}{2k} & \sin\dfrac{\pi}{2k}\\ -\sin\dfrac{\pi}{2k} & \cos\dfrac{\pi}{2k}\end{bmatrix},$$

where

$$1 < \alpha < (\cos\frac{\pi}{2k})^{-1}.$$

(This family appears in the work of Lagarias and Wang [1995] as an example demonstrating that the length of the optimal product cannot be bounded; see Section 3.)

The first construction and its relation to non-algebraicity is presented in Section 2. The second and third constructions are very similar and their relations to the finiteness property are presented in Section 3. One technical difference between the second and third constructions is that it is not known whether the former can produce matrices with *rational* entries, while the latter can do so. In Section 3, we present a result that is of potential interest independent of the above theorem: that existence of a sum of squares Lyapunov function implies the finiteness property of optimal products. This result somehow links lower and upper bound approaches for computation of the joint spectral radius. Similar results were obtained in the pioneering works of Gurvits [1996] for polytopic Lyapunov functions and Lagarias and Wang [1995] for quadratic Lyapunov functions, as well as, several other classes of *convex* Lyapunov functions.

We shall also remark that for *continuous time* switched linear systems, Mason et al. [2006] have established that the degree of a polynomial Lyapunov function for an ASS system may be arbitrarily high, answering a question raised by Dayawansa and Martin. We have been unable to come up with a transformation from continuous time to discrete time that preserves both AAS and non-existence of polynomial Lyapunov functions of any desired degree.

In Section 4, we provide an alternative proof of Theorem 1 based on an undecidability results due to Blondel and Tsitsiklis [2000]. While this will be a non-constructive argument, its implications will be stronger. Indeed, Theorem 1 above implies that the complexity parameter d (and hence the size of underlying LPs and SDPs) cannot be upper bounded as a function of n and m only. The undecidability results, however, imply that d cannot be upper bounded even as a function of n, m, and the entries of the input matrices. We close our paper with some brief concluding remarks in Section 5.

2. COMPLEXITY OF LYAPUNOV FUNCTIONS AND NON-ALGEBRAICITY

One classical approach to demonstrate that a problem is hard is to establish that there is no *algebraic criterion* for testing the property under consideration. This is formalized by showing that the set of instances of a given size that satisfy the property do not form a *semialgebraic set* (see formal definition below). Such a result rules out the possibility of any characterization of the property at hand that only involves operations on the input data that include combinations of arithmetical operations (additions, subtractions, multiplications, and divisions), logical operations ("and" and "or"), and sign test operations (equal to, greater than, greater than or equal to,...); see Blondel and Gevers [1993]. While this is a very strong statement, non-algebraicity does not imply (but is implied by) Turing undecidability, which will be our focus in Section 4. Nevertheless, non-algebraicity results alone are enough to show that the complexity of commonly used Lyapunov functions for switched linear systems cannot be bounded. The goal of this section is to formalize this argument.

Definition 1. *A set $S \subset \mathbb{R}^n$ defined as $S = \{x \in \mathbb{R}^n : f_i(x) \rhd_i 0, i = 1, \ldots, r\}$, where for each i, f_i is a polynomial and $\rhd_i$ is one of $\geq, <, =, \neq$, is called a* basic semialgebraic set*. A set is called* semialgebraic *if it can be expressed as a finite union of basic semialgebraic sets.*

Theorem 2 (Tarski [1951], Seidenberg [1954])**.** *Let $S \subset \mathbb{R}^{k+n}$ be a semialgebraic set and $\pi : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be a projection map that sends $(x, y) \mapsto x$. Then $\pi(S)$ is a semialgebraic set in $\mathbb{R}^n$.*

We start by presenting two examples of semialgebraic sets that are relevant for our purposes.

Lemma 1. *The set $\mathfrak{S}_n$ of stable $n \times n$ real matrices (i.e., those with spectral radius less than one), when viewed as a subset of $\mathbb{R}^{n^2}$, is semialgebraic.*

Proof. An equivalent characterization of stable matrices is via quadratic Lyapunov functions:

$$\begin{aligned}\mathfrak{S}_n = \{A : \ & \exists P = P^T \text{ s.t.}\\ & x^T P x > 0, \forall x \neq 0,\\ & x^T(P - A^T P A)x > 0, \forall x \neq 0\}.\end{aligned} \quad (3)$$

Let $\mathfrak{T}_n$ be the set of $n \times n$ matrices A and P that satisfy these constraints; i.e., P and $P - A^T PA$ both positive definite. The set $\mathfrak{T}_n$ is semialgebraic (in fact basic semialgebraic). One way to see this is to note that the variables x in (3) can be eliminated: Since a matrix is positive definite if and only if its n leading principal minors are positive, we can describe the set $\mathfrak{T}_n$ by $2n$ strict polynomial inequalities (in variables that are the entries of P and A). The set $\mathfrak{S}_n$ is then the projection of $\mathfrak{T}_n$ onto the space of the A variables. Hence, by Theorem 2, $\mathfrak{S}_n$ is semialgebraic. □

Lemma 2. *The set $\mathcal{P}_{n,d}$ of nonnegative polynomials in n variables and (even) degree d is semialgebraic.*

Proof. This is a standard fact in algebra; see e.g. Blekherman et al. [2013]. A polynomial p is by definition nonnegative if

$$\forall x_1 \forall x_2 \ldots \forall x_n \; p(x_1, \ldots, x_n) \geq 0.$$

One can apply quantifier elimination techniques (see e.g. Caviness and Johnson [1998]) to eliminate the quantified variables $(x_1, \ldots, x_n)$ and obtain a description of $\mathcal{P}_{n,d}$ as a semialgebraic set in terms of the coefficients of p only. Another approach is to resort to the representation of nonnegative polynomials as sums of squares of rational functions (à la Hilbert's 17th problem Reznick [2000]) and note the equivalent description:

$$\begin{aligned}\mathcal{P}_{n,d} = \{p : \;& \exists q \in \mathbb{R}[x] \text{ of degree } \hat{d} = \hat{d}(n,d) \text{ s.t.} \\ & q \text{ is sos}, \\ & pq \text{ is sos}\}.\end{aligned}$$

By definition of the sum of squares decomposition, the last two conditions can be written as polynomial equations in the coefficients of the polynomials p, q, and the coefficients of the new polynomials that are being squared. Hence, the result is a semialgebraic set. The set $\mathcal{P}_{n,d}$ is the projection of this set onto the space of coefficients of p and is therefore semialgebraic. □

Unlike the case of stable matrices (Lemma 1), when we move to switched systems defined by even only two matrices, the set of stable systems no longer defines a semialgebraic set. This is a result of Kozyakin [1990]. The result is stated in terms of the *joint spectral radius* (see Jungers [2009] for a monograph on the topic) which captures the stability of a linear switched system.

Definition 2. *(Rota and Strang [1960]) If $\|\cdot\|$ is any matrix norm, consider $\rho_k(\Sigma) := \sup_{A_i \in \Sigma} \|A_1 \ldots A_k\|^{1/k}, \quad k \in \mathbb{N}$. The* joint spectral radius *(JSR) of $\mathcal{M}$ is*

$$\rho(\Sigma) = \lim_{k \to \infty} \rho_k(\Sigma). \tag{4}$$

The joint spectral radius does not depend on the matrix norm chosen thanks to the equivalence between matrix norms in finite dimensional spaces. It is well known that the switched system in (1) is absolutely asymptotically stable if and only if $\rho(\Sigma) < 1$.

Theorem 3 (Kozyakin [1990]; see also Theys [2005])**.** *The set of 2×2 matrices A_1, A_2 with $\rho(A_1, A_2) < 1$ is not semialgebraic.*

Proof. The proof of Kozyakin is established by showing that the family of matrices

$$A_1 = \frac{(1-t^4)}{(1-3\pi t^3/2)} \begin{bmatrix} \sqrt{1-t^2} & -t \\ 0 & 0 \end{bmatrix},$$

$$A_2 = (1-t^4) \begin{bmatrix} \sqrt{1-t^2} & -t \\ t & \sqrt{1-t^2} \end{bmatrix},$$

have JSR less than one for $t = \sin \frac{2\pi}{2k+1}$ (for $k \in \mathbb{N}$ large enough), and JSR more than one for $t = \sin \frac{2\pi}{2k}$ (for $k \in \mathbb{N}$ large enough). Hence, the stability set has an infinite number of disconnected components and therefore cannot be semialgebraic. □

We now show that by contrast, for any integer d, the set of matrices $\{A_1, \ldots, A_m\}$ that admit a common polynomial Lyapunov function of degree $\leq d$ is in fact semialgebraic. This establishes the result related to the first construction in Theorem 1.

Theorem 4. *For any positive integer d, the set of matrices $\{A_1, \ldots, A_m\}$ (viewed as a subset of $\mathbb{R}^{mn^2}$) that admit either (i) a polynomial Lyapunov function of degree $\leq d$, or (ii) a polytopic Lyapunov function with $\leq d$ facets, or (iii) a piecewise quadratic Lyapunov function (in form of max-of-quadratics or min-of-quadratics) with $\leq d$ pieces is semialgebraic.*

Proof. We prove the claim only for polynomial Lyapunov functions. The proof of the other claims are very similar and will appear in an expanded version of this paper. The goal is to show that the set

$$\begin{aligned}\mathfrak{T}_{n,m} := \{A_1, \ldots, A_m : \;& \exists p := p(x),\ degree(p) \leq d, \text{ s.t.} \\ & p(x) > 0, \forall x \neq 0, \\ & p(x) - p(A_i x) > 0, \forall x \neq 0, \\ & \forall i = 1, \ldots, m\}\end{aligned} \tag{5}$$

is semialgebraic. Our argument is a simple generalization of that of Lemma 1 and relies on Lemma 2. It follows from the latter lemma that for each $\hat{d} \in \{2, \ldots, d\}$, the set of polynomials p of degree $\hat{d}$ and matrices $\{A_1, \ldots, A_m\}$ that together satisfy the constraints in (5) is semialgebraic. (Note that these are sets in the coefficients of p and the entries of the matrices A_i, as the variables x are being eliminated.) The union of these sets is of course also semialgebraic. By the Tarski–Seidenberg theorem (Theorem 2), once we project the union set onto the space of variables in A_i, we still obtain a semialgebraic set. Hence, $\mathfrak{T}_{n,m}$ is semialgebraic. □

3. COMPLEXITY OF LYAPUNOV FUNCTIONS AND THE FINITENESS PROPERTY OF OPTIMAL PRODUCTS

A set of matrices $\{A_1, \ldots, A_m\}$ satisfies the *finiteness property* if its JSR is achieved as the spectral radius of a finite product; i.e., if

$$\rho(A_1, \ldots, A_m) = \rho^{1/k}(A_{\sigma_k} \ldots A_{\sigma_1}),$$

for some k and some $(\sigma_k, \ldots, \sigma_1) \in \{1, \ldots, m\}^k$. The matrix product $A_{\sigma_k} \ldots A_{\sigma_1}$ that achieves the JSR is called the *optimal product* and generates the "worst case trajectory" of the switched system in (1). The finiteness conjecture of Lagarias and Wang [1995] (see also Gurvits [1992], where the conjecture is attributed to Pyatnitskii) asserts that all sets of matrices have the finiteness property. The conjecture was disproved in 2002 by Bousch and Mairesse [2002] with alternative proofs consequently appearing in Blondel et al. [2003], Kozyakin [2005], and Hare et al. [2011]. In particular, the last reference provided the first explicit counterexample only recently. It is currently not known whether all sets of matrices with *rational* entries satisfy the finiteness property (Jungers and Blondel [2008a]).

Gurvits [1992] shows that if the set of matrices admits a polytopic Lyapunov function, then the finiteness property holds. The result is generalized by Lagarias and Wang [1995] to Lyapunov functions that take the form of various other norms, including ellipsoidal norms. In this section, we combine the result of Lagarias and Wang on ellipsoidal norms with some algebraic lifting arguments to establish that sets of matrices which admit a sum of squares (sos) Lyapunov function always satisfy the finiteness property. Note that sos Lyapunov functions of degree ≥ 4 do not in general define a norm as their sublevel sets may very well be non-convex. Similar arguments imply that existence of a piecewise quadratic Lyapunov function also results in the finiteness property, though we restrict our attention here to sos Lyapunov functions.

Theorem 5 (Lagarias and Wang [1995])**.** *The finiteness property holds for any set of $n \times n$ matrices $\{A_1, \ldots, A_m\}$ of JSR equal to one that share an ellipsoidal norm, i.e., satisfy $A_i^T P A_i \preceq P$ for some symmetric positive definite matrix P. Moreover, the length of the optimal product is upper bounded by a quantity that depends on n and m only.* [5]

Theorem 6. *Let $\{A_1, \ldots, A_m\}$ be a set of $n \times n$ matrices of JSR equal to one. If there exists a (homogeneous) positive definite polynomial p of degree $2d$ that satisfies*

$$p(x) \ sos, \ p(x) - p(A_i x) \ sos, i = 1, \ldots, m,$$

then $\{A_1, \ldots, A_m\}$ satisfies the finiteness property. Moreover, the length of the optimal product is upper bounded by a quantity that depends on n, m, and d only.

Proof. Let p be a polynomial Lyapunov function of degree $2d$. Since $p(x)$ and $p(x) - p(A_i x)$ are sos, there exist symmetric matrices $Q_0, Q_1, \ldots, Q_m$ of size $\binom{n+d-1}{d}$ such that

$$\begin{aligned} p(x) &= x^{[d]^T} Q_0 x^{[d]} \\ p(x) - p(A_i x) &= x^{[d]^T} Q_i x^{[d]}, \ i = 1, \ldots, m \\ Q_0, Q_1, \ldots, Q_m &\succeq 0, \end{aligned} \tag{6}$$

where $x^{[d]^T}$ is the d-lift of the vector x, a vector of scaled monomials of degree exactly d with components $\{\sqrt{\gamma! x^\gamma}\}_\gamma$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, $\sum_i \gamma_i = d$, and $\gamma!$ indicates the multinomial coefficient $\gamma! := \frac{d!}{\gamma_1! \gamma_1! \ldots \gamma_1 n!}$. We show that the constraints in (6) imply existence of a common quadratic Lyapunov function in a higher dimension for a lifted set of matrices $\{A_1^{[d]}, \ldots, A_m^{[d]}\}$ of size $\binom{n+d-1}{d} \times \binom{n+d-1}{d}$. These matrices are given by the *Veronese lifting*:

$$(A^{[d]})_{\alpha\beta} = \frac{\mathrm{per} A(\alpha, \beta)}{\sqrt{\eta(\alpha)\eta(\beta)}},$$

where the indices α, β are all the d-element multisets of $\{1, \ldots, n\}$ (there is exactly $\binom{n+d-1}{d}$ of them). The notation per denotes the permanent of a matrix, and $\eta(S)$ is a scalar obtained by multiplying the factorials of the multiplicities of the elements of a multiset S. See Parrilo and Jadbabaie [2008] for more details and an explicit example of this construction. One can show the following nice matrix-vector and matrix-matrix multiplication properties of this lifting

$$(Ax)^{[d]} = A^{[d]} x^{[d]}, \tag{7}$$

$$(AB)^{[d]} = A^{[d]} B^{[d]}. \tag{8}$$

Moreover, for any matrix B the spectral radius of the lifted matrix satisfies

$$\rho(B^{[d]}) = \rho^d(B). \tag{9}$$

From (6) and (7) it follows that

$$x^{[d]^T} Q_0 x^{[d]} - x^{[d]^T} A_i^{[d]^T} Q_0 A_i^{[d]} x^{[d]} = x^{[d]^T} Q_i x^{[d]}.$$

Hence $Q_0 - A_i^{[d]^T} Q_0 A_i^{[d]}$ is positive semidefinite for all $i = 1, \ldots, m$. But this precisely means that $x^{[d]^T} Q_0 x^{[d]}$ is a quadratic Lyapunov function for the lifted set of matrices $\{A_1^{[d]}, \ldots, A_m^{[d]}\}$. By Theorem 5, the set $\{A_1^{[d]}, \ldots, A_m^{[d]}\}$ satisfies the finiteness property and the length k of its optimal product is bounded by a quantity depending only on m and $\binom{n+d-1}{d}$. Let us denote this optimal product by

$$A_{\sigma_k}^{[d]} \ldots A_{\sigma_1}^{[d]},$$

where $\sigma_k \in \{1, \ldots, m\}$. We claim that the matrix product

$$A_{\sigma_k} \ldots A_{\sigma_1}$$

should be an optimal product for the original set of matrices $\{A_1, \ldots, A_m\}$. This is true because for any matrix product given by some arbitrary indices $\bar{\sigma}_{\bar{k}}, \ldots, \bar{\sigma}_1 \in \{1, \ldots, m\}^{\bar{k}}$, (8) and (9) imply the relations

$$\rho(A_{\bar{\sigma}_{\bar{k}}}^{[d]} \ldots A_{\bar{\sigma}_1}^{[d]}) = \rho((A_{\bar{\sigma}_{\bar{k}}} \ldots A_{\bar{\sigma}_1})^{[d]}) = \rho^d(A_{\bar{\sigma}_{\bar{k}}} \ldots A_{\bar{\sigma}_1}).$$

□

We conjecture that the assumption of having a sum of squares Lyapunov function in our Theorem 6 can be weakened to the assumption of having a polynomial Lyapunov function. In dimension two, these two classes of Lyapunov functions are the same (see Lemma 3 below) and this allows us to show that any family of 2×2 matrices for which the length of the optimal product blows up is also a family of matrices where the degree of a stability proving polynomial Lyapunov function is forced to blow up. This is the idea behind constructions 1 and 3 in Theorem 1.

Lemma 3. *For switched linear systems in two variables, the set of polynomial Lyapunov functions of any degree d coincides with the set of sum of squares Lyapunov functions of degree d.*

Proof. It is a standard fact that nonnegative homogeneous polynomials in 2 variables are sums of squares Reznick [2000]. The Lyapunov inequalities under consideration are $p(x) > 0$ and $p(x) - p(A_i x) > 0$, where $x = (x_1, x_2)^T$. These are both homogeneous polynomial inequalities in two variables. □

Corollary 1. *Let $A_1, \ldots, A_m$ be any set of 2×2 matrices with JSR 1 that violate the finiteness property. An example is*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ A_2 = \alpha_* \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

[5] An expression for this bound is presented in (Lagarias and Wang [1995]), which can easily be extended to our generalization.

with

$\alpha_* \simeq 0.749326546330367557943961948091344672091...$[6] .

For any positive integer d, there exists a positive integer k such that the set of matrices $(1-\frac{1}{k})\{A_1,\dots,A_m\}$ is asymptotically stable under arbitrary switching but does not admit a polynomial Lyapunov function of degree $\leq d$.

Proof. For all $k>1$, the matrices $(1-\frac{1}{k})\{A_1,\dots,A_m\}$ have JSR less than one and therefore are asymptotically stable under arbitrary switching. Suppose for the sake of contradiction that there exists a d such that for all $k>0$ the matrices have a polynomial Lyapunov function of degree $\leq d$. By Lemma 3, the matrices would also have a sum of squares Lyapunov function of degree $\leq d$. But then Theorem 6 would imply that (for all k) the matrices have the finiteness property with an optimal product whose length is upper bounded by a fixed quantity. This contradicts the assumption that A_1 and A_2 do not satisfy the finiteness property. □

The next corollary is very similar but the matrix family that it presents is completely explicit.

Corollary 2. *Consider the matrix family $(1-\frac{1}{k})\{A_1,A_2\}$, with*

$$A_1=\alpha^k\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ A_2=\alpha^{-1}\begin{bmatrix}\cos\frac{\pi}{2k} & \sin\frac{\pi}{2k}\\ -\sin\frac{\pi}{2k} & \cos\frac{\pi}{2k}\end{bmatrix},$$

where

$$1<\alpha<(\cos\frac{\pi}{2k})^{-1}.$$

For any positive integer d, there exists a positive integer k such that the set of matrices $(1-\frac{1}{k})\{A_1,A_2\}$ is asymptotically stable under arbitrary switching but does not admit a polynomial Lyapunov function of degree $\leq d$.

Proof. Consider the matrix family $\{A_1(k),A_2(k)\}$. Lagarias and Wang [1995] show that the JSR of this family is 1 for all k. On the other hand, they show that the spectral radii of all products of length $\leq k$ are less than one, whereas there is a product of length $k+1$ which achieves the JSR; i.e., has spectral radius one. As a result, by increasing k, the length of the optimal product blows up. Hence, by Lemma 3 and Theorem 6, the degree of a polynomial Lyapunov function (applied to the asymptotically stable family $(1-\frac{1}{k})\{A_1(k),A_2(k)\}$) must blow up as well. □

4. COMPLEXITY OF LYAPUNOV FUNCTIONS AND UNDECIDABILITY

In this section, we show that our statements on lack of upper bounds on complexity of Lyapunov functions also follow in a straightforward manner from undecidability results. Compared to the results of the previous sections, the new statements are weaker in some sense and stronger in some other. They are weaker in that the statements are *non-constructive.* However, they imply the stronger statement that the complexity of Lyapunov functions (e.g., degree or number of pieces) cannot be upper bounded, not only as a function of n and m, but also as a computable function of n, m, and the entries of the matrices in Σ (Corollary 4). In addition to this, we can further establish that the same statements are true for very simple and restricted classes of matrices whose entries take two different values only (see Theorem 10).

Theorem 7. *For any positive integer d, there are families of matrices of size 47×47 that are asymptotically stable under arbitrary switching but do not admit (i) a polynomial Lyapunov function of degree d, or (ii) a polytopic Lyapunov function with d facets, or (iii) a max-of-quadratics Lyapunov function with d pieces, or (iv) a min-of-quadratics Lyapunov function with d pieces.*

The main ingredient in the proof is the following undecidability theorem, which is stated in terms of the JSR of a set of matrices.

Theorem 8. *(Blondel and Canterini [2003], Blondel and Tsitsiklis [2000]) The problem of determining, given a set of matrices Σ, if $\rho(\Sigma)\leq 1$ is Turing-undecidable. This result remains true even if Σ contains only two matrices with nonnegative rational entries of size 47×47.*

We now show that this result implies Theorem 7. The main ingredient is Tarski's quantifier elimination theory, which gives a finite time procedure for checking certain quantified polynomial inequalities. The rest is a technical transformation of the problem "$\rho\leq 1$?" to the existence of a degree d polynomial Lyapunov function.

Proof of theorem 7. We suppose by contradiction that every AAS set of matrices of size 47×47 has a polynomial Lyapunov function of degree at most d, for some even natural number d. We claim that this implies the algorithmic decidability of the question "$\rho\leq 1$?".

Indeed, by homogeneity of the JSR, we have

$$\rho(\Sigma)\leq 1 \Leftrightarrow \forall\epsilon\in(0,1),\ \rho((1-\epsilon)\Sigma)<1.$$

Now, by the hypothesis above, the last statement should be equivalent to the existence of a polynomial Lyapunov function of degree less than d for $(1-\epsilon)\Sigma$, and thus we can rephrase the question $\rho(\Sigma)\leq 1$ as follows. (In what follows, $\mathcal{P}^+_{n,d}$ is the set of polynomials in n variables and (even) degree d whose value is positive on $\mathbb{R}^n\setminus\{0\}$. This set is semi-algebraic. See Lemma 2 for a similar statement.)

$$\forall\epsilon\in(0,1),\ \exists p(\cdot)\in\mathcal{P}^+_{n,d}\text{ such that}$$
$$\forall x\in\mathbb{R}^{47}\setminus\{0\},\ \forall A\in\Sigma,\quad p((1-\epsilon)Ax)<p(x).$$

Since the assertion above is semi-algebraic, it is possible to algorithmically decide whether it is true or not. Thus, it allows to decide whether $\rho(\Sigma)\leq 1$, contradicting Theorem 8.

The proof for the polytopic Lyapunov function and max/min-of-quadratics goes exactly the same way, by noticing that once their number of components is fixed, one can decide their existence by Tarski Quantifier Elimination as well. □

In Blondel and Tsitsiklis [2000], the authors note that Theorem 8 implies the following result:

Corollary 3. *(Blondel and Tsitsiklis [2000]) There is no effectively computable function*[7] *$t(\Sigma)$, which takes an*

[6] See Hare et al. [2011] for an expression for the exact value of α_*

[7] See (Blondel and Tsitsiklis [2000]) for a definition.

arbitrary set of matrices with rational entries Σ, *and returns in finite time a natural number such that*

$$\rho(\Sigma) = \max_{t' \leq t(\Sigma)} \max_{A \in \Sigma} \rho(A).$$

The same corollary can be derived concerning the degree of a Lyapunov function.

Corollary 4. *There is no effectively computable function* $d(\Sigma)$, *which takes an arbitrary set of matrices with rational entries* Σ, *and returns in finite time a natural number such that if* $\rho(\Sigma) < 1$, *there exists a polynomial Lyapunov function of degree less than* d.

Next, we show a similar result, which does not focus on the fixed size of the matrices in the family, but somehow on the complexity of the real numbers defining the entries of the matrices. Namely, we show that such negative results also hold essentially for sets of binary matrices (that is, matrices with only 0/1 entries). In fact, the very question $\rho \leq 1$ is easy to answer in this case (see Jungers et al. [2008]), so, one cannot hope to have strong negative results stated in terms of binary matrices. However, it turns out that for an arbitrary integer K the question $\rho \leq K$ for binary matrices is as hard as the question $\rho \leq 1$ for rational matrices. More precisely, we have the following theorem:

Theorem 9. *(Jungers and Blondel [2008b]) Given a set of* m *nonnegative rational matrices* Σ, *it is possible to build a set of* m *binary matrices* Σ' *(possibly of larger dimension), together with a natural number* K *such that for any product* $A = A_{i_1} \ldots A_{i_t} \in \Sigma^t$, *the corresponding product* $A'_{i_1} \ldots A'_{i_t} \in \Sigma'^t$ *has numerical values in its entries that are exactly equal to zero, or to entries in the product* A *multiplied by* K^t. *Moreover, for any entry in the product* A, *there is an entry in the product* A' *with the same value multiplied by* K^t.

Theorem 8 together with Theorem 9 allows us to prove another negative result on the degree of Lyapunov functions restricted to matrices with entries all equal to a same number $1/K$, $K \in \mathbb{Q}$. Remark that the fact that the parameter $K \in \mathbb{Q}$ has unbounded denominator and numerator is unavoidable in such an undecidability theorem, since for bounded values, there is a finite number of matrices with all entries in the range, and this rules out a result as the one in the theorem below.

Theorem 10. *There is no function* $d : \mathbb{N} \to \mathbb{N}$ *such that for any set of matrices of dimension* n *with entries all equal to a same number* 1 *or* K, $K \in \mathbb{Q}$, *the set is AAS if and only if there exists a (strict) polynomial Lyapunov function of degree* $d(n)$.

(sketch). Given a set of rational matrices of size 47×47, Theorem 9 allows us to build a set of binary $n' \times n'$ matrices Σ' such that $\rho(\Sigma') = K\rho(\Sigma)$. Thus, $\rho(\Sigma'/K) = \rho(\Sigma)$, and the existence of strict polynomial Lyapunov functions for AAS $n' \times n'$ matrices would again imply decidability of the question $\rho \leq 1$ for Σ'/K, which again contradicts Theorem 8. □

The point of our last theorem was to show that in high dimensions, "bad" families of matrices that necessitate arbitrarily complex Lyapunov functions can have very simple and structured entries.

5. CONCLUSION

In this paper, we leveraged results related to non-algebraicity, undecidability, and the finiteness property of the joint spectral radius to demonstrate that commonly used Lyapunov functions for switched linear systems can be arbitrarily complex, even in fixed dimension, or for matrices with lots of structure.

If these negative results are bad news for the practitioner, it is worth mentioning that in practice the different Lyapunov functions often have complementary performance. So while there certainly exist instances which make all methods fail (as we have shown), one can hope that in practice, at least one of the different Lyapunov methods would be able to certify stability. In light of this, we believe it is important to (i) understand systematically how the different methods compare to each other, and (ii) identify subclasses of matrices that if stable, are guaranteed to admit "simple" Lyapunov functions. While the latter objective has been reasonably achieved for quadratic Lyapunov functions, results of similar nature are lacking for even slightly more complicated Lyapunov functions (say, polynomials of degree 4, or piecewise quadratics with 2 pieces).

REFERENCES

A. A. Ahmadi. On the difficulty of deciding asymptotic stability of cubic homogeneous vector fields. In *Proceedings of the American Control Conference*, 2012.

A. A. Ahmadi and R. Jungers. SOS-convex Lyapunov functions with applications to nonlinear switched systems. In *Proceedings of the IEEE Conference on Decision and Control*, 2013.

A. A. Ahmadi, R. Jungers, P. A. Parrilo, and M. Roozbehani. Joint spectral radius and path-complete graph Lyapunov functions. *SIAM Journal on Optimization and Control*, 2013. To appear.

T. Ando and M.-H. Shih. Simultaneous contractibility. *SIAM Journal on Matrix Analysis and Applications*, 19: 487–498, 1998.

G. Blekherman, P. A. Parrilo, and R. Thomas. *Semidefinite optimization and convex algebraic geometry.* SIAM Series on Optimization, 2013.

V. Blondel and M. Gevers. Simultaneous stabilizability of three linear systems is rationally undecidable. *Mathematics of Control, Signals and Systems*, 6(2):135–145, 1993.

V. D. Blondel and V. Canterini. Undecidable problems for probabilistic automata of fixed dimension. *Theory of Computing Systems*, 36(3):231–245, 2003.

V. D. Blondel and J. N. Tsitsiklis. The boundedness of all products of a pair of matrices is undecidable. *Systems and Control Letters*, 41:135–140, 2000.

V. D. Blondel, J. Theys, and A. A. Vladimirov. An elementary counterexample to the finiteness conjecture. *SIAM Journal on Matrix Analysis and Applications*, 24 (4):963–970, 2003.

T. Bousch and J. Mairesse. Asymptotic height optimization for topical IFS, Tetris heaps, and the finiteness conjecture. *Journal of the American Mathematical Society*, 15(1):77–111, 2002.

B. F Caviness and J. R. Johnson. *Quantifier elimination and cylindrical algebraic decomposition*. Springer Verlag, 1998.

W. P. Dayawansa and C. F Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, 44(4):751–760, 1999.

R. Goebel, A. R. Teel, T. Hu, and Z. Lin. Conjugate convex Lyapunov functions for dual linear differential inclusions. *IEEE Transactions on Automatic Control*, 51(4):661–666, 2006.

L. Gurvits. Stability of discrete linear inclusions. *Linear Algebra Appl.*, 231:47–85, 1992.

L. Gurvits. Stability of linear inclusions - part 2. *NECI technical report TR*, pages 96–173, 1996.

K. G. Hare, I. D. Morris, N. Sidorov, and J. Theys. An explicit counterexample to the Lagarias–Wang finiteness conjecture. *Advances in Mathematics*, 226(6):4667–4701, 2011.

R. Jungers. *The joint spectral radius: theory and applications*, volume 385 of *Lecture Notes in Control and Information Sciences*. Springer, 2009.

R. M. Jungers and V. D. Blondel. On the finiteness property for rational matrices. *Linear Algebra and its Applications*, 428(10):2283–2295, 2008a.

R. M. Jungers and V. D. Blondel. On the finiteness property for rational matrices. *Linear Algebra and its Applications*, 428(10):2283–2295, 2008b. doi: 10.1016/j.laa.2007.07.007. URL `http://arxiv.org/abs/math/0702489`.

R. M. Jungers, V. Yu. Protasov, and V. D. Blondel. Efficient algorithms for deciding the type of growth of products of integer matrices. *Linear Algebra and its Applications*, 428(10):2296–2311, 2008. doi: 10.1016/j.laa.2007.08.001. URL `http://www.citebase.org/abstract?id=oai:arXiv.org:cs/0604047`.

V. Kozyakin. A dynamical systems construction of a counterexample to the finiteness conjecture. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 2338–2343. IEEE, 2005.

V. A. Kozyakin. Algebraic unsolvability of problem of absolute stability of desynchronized systems. *Automation and Remote Control*, 51:754–759, 1990.

J. C. Lagarias and Y. Wang. The finiteness conjecture for the generalized spectral radius of a set of matrices. *Linear Algebra and its Applications*, 214:17–42, 1995.

D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, MA, 2003.

H. Lin and P. J. Antsaklis. Stability and stabilizability of switched linear systems: a short survey of recent results. In *Proceedings of IEEE International Symposium on Intelligent Control*, 2005.

O. Mason and R. Shorten. On common quadratic Lyapunov functions for stable discrete time LTI systems. *IMA Journal of Applied Maths*, 59:271–283, 2004.

P. Mason, U. Boscain, and Y. Chitour. Common polynomial Lyapunov functions for linear switched systems. *SIAM Journal on Optimization and Control*, 45(1), 2006.

A. Olshevsky and J. N. Tsitsiklis. On the nonexistence of quadratic Lyapunov functions for consensus algorithms. *IEEE Transactions on Automatic Control*, 53(11):2642–2645, 2008.

P. A. Parrilo and A. Jadbabaie. Approximation of the joint spectral radius using sum of squares. *Linear Algebra Appl.*, 428(10):2385–2402, 2008.

A. Polański. Lyapunov function construction by linear programming. *IEEE Transactions on Automatic Control*, 42(7):1013–1016, 1997.

A. Polański. On absolute stability analysis by polyhedral Lyapunov functions. *Automatica*, 36(4):573–578, 2000.

B. Reznick. Some concrete aspects of Hilbert's 17th problem. In *Contemporary Mathematics*, volume 253, pages 251–272. American Mathematical Society, 2000.

G. C. Rota and W. G. Strang. A note on the joint spectral radius. *Indag. Math.*, 22:379–381, 1960.

A. Seidenberg. A new decision method for elementary algebra. *Ann. of Math. (2)*, 60:365–374, 1954.

R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49(4):545–592, 2007. doi: 10.1137/05063516X. URL `http://epubs.siam.org/doi/abs/10.1137/05063516X`.

A. Tarski. *A decision method for elementary algebra and geometry*. University of California Press, Berkeley and Los Angeles, Calif., 1951. Second edition.

J. Theys. *Joint Spectral Radius: theory and approximations*. PhD thesis, PhD thesis, Universite catholique de Louvain, 2005.