RC25451 (WAT1402-081) February 27, 2014
Mathematics

IBM Research Report

Quadratic Systems Do Not Have Algebraic Limit Cycles of Degree 3

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QUADRATIC SYSTEMS DO NOT HAVE ALGEBRAIC LIMIT CYCLES OF DEGREE 3

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Abstract. To be written.

2000 AMS Subject Classification : 34C05.
Keywords and phrases: Invariant algebraic curves, quadratic systems, algebraic limit cycles.

1. Introduction and statement of the main results

We shall study polynomial vector fields in $I\!R^2$ defined by systems

$$\begin{aligned} \dot{x} &= p(x, y), \\ \dot{y} &= q(x, y), \end{aligned} \tag{1}$$

where p, q are coprime polynomials of degree 2, i.e.

$$p(x, y) = \sum_{i,j=0}^{2} p_{i,j} x^i y^j, \qquad q(x, y) = \sum_{i,j=0}^{2} q_{i,j} x^i y^j$$

We shall call these systems *quadratic systems*.

The object of our study will be the limit cycles of such systems, mainly the algebraic ones, i.e. the limit cycles contained in the zero set of some polynomial

$$\varphi(x, y) = \sum_{i,j=0}^{n} = \varphi_{i,j} x^i y^j.$$

It is well–known that each limit cycle of a polynomial vector field must surround at least one critical point, and for a quadratic system inside each limit cycle there must be precisely one critical point of focus type, see [**?**].

The algebraic curve $\varphi(x, y) = 0$ is an invariant algebraic curve of system (1) if and only if there exists a polynomial $\kappa = \kappa(x, y)$ satisfying

$$p\frac{\partial \varphi}{\partial x} + q\frac{\partial \varphi}{\partial y} - \kappa\varphi = 0. \tag{2}$$

The polynomial κ is called a *cofactor* of the curve $\varphi = 0$. In case of quadratic systems the degree of the cofactor can be at most 1. An invariant algebraic curve $\varphi = 0$ is called *irreducible* if the polynomial φ is irreducible.

A trajectory γ of system (1) is a *limit cycle* if it is homeomorphic to a circle and there are no other periodic trajectories in some neighborhood of γ. The orbit γ is an *algebraic limit cycle* of system (1) if it is a limit cycle and it is contained in some irreducible algebraic invariant curve $\varphi = 0$ of system (1). The *degree* of an algebraic limit cycle γ is the degree of φ.

2. Preliminaries.

We shall call the point (x, y) a *critical point of the system (1)* if and only if $p(x, y) = q(x, y) = 0$. We shall call the point (x, y) a *critical point of a function* φ if and only if $\frac{\partial\varphi}{\partial x}(x, y) = \frac{\partial\varphi}{\partial y}(x, y) = 0$.

Now immediately from the definitions it follows

Proposition 1. *All the critical points of system (1) and all the critical points of* φ *are contained in the union of sets* $\{\kappa = 0\} \cup \{\varphi = 0\}$.

3. Quadratic systems have no degree 3 algebraic limit cycles

Definition 2. *The Milnor number of a germ of an analytic function* $f \in \mathcal{O}_p$ *at* $p \in \mathbb{C}$ *is defined as*

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Lemma 3. *Let* $\varphi(x, y) = 0$ *be an algebraic curve of degree* 3*. If there exists* p *such that the Milnor number of* φ $\mu_p(\varphi) > 2$ *then either* $\varphi = 0$ *contains no real ovals or in an appropriate system of affine coordinates* $\varphi = c + x^2 + Ax^3 + xy^2$

Proof. Without loss of generality we may assume that $x_0 = y_0 = 0$. If curve $\varphi = 0$ contains an oval, the point p cannot be a triple point of φ, thus for an appropriate choice of affine coordinates $\varphi = c + x^2 + axy + Ax^3 + Bx^2y + Cxy^2 + Dy^3$. If $a \neq 0$ then $\frac{\partial\varphi}{\partial x} = 2x + ay + \ldots$, $\frac{\partial\varphi}{\partial y} = ax + \ldots$ and $\mu_p(\varphi) = 1$. If $a = 0$ then if $D \neq 0$ then $\frac{\partial\varphi}{\partial x} = 2x + \ldots$, $\frac{\partial\varphi}{\partial y} = 2Dy^2 + xg_1(x, y)$ and $\mu_p(\varphi) = 2$. If $D = C = 0$ then φ is linear in y and $\varphi = 0$ contains no ovals. If $C \neq 0$ then an affine change of coordinate y transforms φ to the form $\varphi = c + x^2 + Ax^3 + xy^2$. □

Lemma 4. *If a cubic algebraic curve* $\varphi(x, y) = 0$ *has a real oval* γ *then there exists a point* p *inside* γ *such that* $\mu_p(\varphi) = 1$

Proof. The region bounded by γ is a compact set, so φ has a local extremum inside it. Without loss of generality we can assume that this point is $(0, 0)$ and that φ has a minimum there. Because $(0, 0)$ is a local minimum, $\nabla\varphi|_{(0,0)} = (0, 0)$ and $D^2\varphi(0, 0)$ must be either positive-determinate or semi-positive determinate. In the first case for an appropriate choice of affine change of coordinates (u, v) there is $\varphi(u, v) = \varphi(0, 0) + au^2 + bv^2 + \ldots$, $ab > 0$ and the Lemma 4 follows. If $D^2\varphi(0, 0)$ is semi-positive determinate then $\varphi(u, v) = \varphi(0, 0) + au^2 + Au^3 + Bu^2v + Cuv^2 + Dv^3$, $a \geq 0$. We have $\varphi(0, v) = Dv^3$. If there was $D \neq 0$ the point $(0, 0)$ would not be a local minimum, so $D = 0$. But then $\varphi(0, v) \equiv \varphi(0, 0)$ and the point $(0, 0)$ could not be inside the oval $\varphi(u, v) = 0$. □

Theorem 5. *A quadratic system cannot have an algebraic limit cycle of degree* 3*.*

Proof. For the cubic curve $\varphi(x, y) = 0$ to have an oval it has to be smooth. Therefore $\varphi = \frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0$ has no solutions. Thus by Proposition 1 all points with $\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0$ are contained in the set $\kappa = 0$. We have three possibilities:

(i) $\deg\frac{\partial\varphi}{\partial x} = \deg\frac{\partial\varphi}{\partial y} = 2$ and $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ have no common factors. By Bézout's theorem $I\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = 4$. On the other hand either at least one of the numbers $I\left(\kappa, \frac{\partial\varphi}{\partial x}\right)$, $I\left(\kappa, \frac{\partial\varphi}{\partial y}\right)$ is equal to 2, or κ is nonzero constant or zero, or κ, $\frac{\partial\varphi}{\partial x}$ and

$\frac{\partial\varphi}{\partial y}$ have a common factor. The last case cannot happen because $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ have no common factors. If $\kappa = \text{const}$ then the set $\kappa = 0$ is an $\emptyset$ so it cannot contain critical points of φ. If $\kappa \equiv 0$ then φ is a first integral and the quadratic system (1) is hamiltonian, so it has no limit cycles. Thus the cardinality of the set of the critical points of φ critφ can be either 1 or 2. By definition $I\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = \sum_p \mu_p(\varphi)$. By Lemma 4 there exists a critical point p of φ with $\mu_p(\varphi) = 1$, so the only possibility we have is that there exists point $q \in \text{crit}\varphi$, $q \neq p$ with $\mu_q(\varphi) = 3$. Then by Lemma 3 either $\varphi = 0$ contains no real ovals or $\varphi = c + x^2 + Ax^3 + xy^2$. In this case both singular points of φ are on the line $y = 0$ and the cofactor must satisfy $\kappa(x, y) = Ky$. We have $p(x, y)(y^2 + 3Ax^2 + 2x) + q(x, y)2xy = Ky(c + x^2 + Ax^3 + xy^2)$, so either $c = 0$ and φ is divisible by x, or $K = 0$ and the system has a polynomial first integral, in either case $\varphi = 0$ cannot contain limit cycles.

(ii) $\deg\frac{\partial\varphi}{\partial x} = \deg\frac{\partial\varphi}{\partial y} = 2$ and $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ have a common factor g. If $\deg g = 2$ then $\frac{\partial\varphi}{\partial x} = 0$ would be a quadratic curve of singular points of φ and it could not be contained in κ. Thus $\deg g = 1$ and without loss of generality we can assume that $g(x, y) = y$. We have then $\frac{\partial\varphi}{\partial x} = ya(x, y)$, $\frac{\partial\varphi}{\partial y} = yb(x, y)$ and $a + y\frac{\partial a}{\partial y} = y\frac{\partial b}{\partial x}$, so $a = cy$ and $b = 2cx + d$. Thus $\varphi(x, y) = cxy^2 + A_3(y) = cxy^2 + dy + B_3(x)$ therefore $\varphi(x, y) = cxy^2 + dy + e$, so $\varphi = 0$ does not contain any ovals.

(iii) One of $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ is linear. Without loss of generality we can assume that $\frac{\partial\varphi}{\partial y} = 2y$. As the two (counting multiplicities) intersection points of the line $\kappa = 0$ coincide with the two (counting multiplicities) intersection points of $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ it follows that $\kappa = 2Ky$ for some $K \neq 0$. We get $p(x, y)\frac{\partial\varphi}{\partial x} + q(x, y)2y = 2Ky\varphi(x, y)$, so $p(x, y)$ must be divisible by y, $p(x, y) = 2(ax + by + c)y$. From $\frac{\partial\varphi}{\partial y} = 2y$ follows $\varphi(x, y) = y^2 + g_3(x)$ and we get $(ax + by + c)g_3'(x) + q(x, y) = K(y^2 + g_3(x))$. The product $byg_3'(x)$ is the only expression containing the term x^2y, so b must be equal to 0. We get $p(x, y) = 2(ax+c)y$, $a \neq 0$ and $q(x, y) = K(y^2+g_3(x))-(ax+c)g_3'(x) = Ky^2+h_3(x)$. But the vector field $2(ax+c)y\frac{\partial}{\partial x}+(Ky^2+h_3(x))\frac{\partial}{\partial y}$ has an integrating factor $(ax + c)^{-\frac{a+K}{a}}$ and a first integral $\left(x^3 - 10x + y^2 - 1\right)(ax + c)^{-\frac{K}{a}}$, so it has no limit cycles.

□

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