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## The Dominating Set Polytope via Facility Location

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# The dominating set polytope via facility location

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**Abstract.** In this paper we present an extended formulation for the dominating set polytope via facility location. We show that with this formulation we can describe the dominating set polytope for cacti graphs, though its description in the natural node variables dimension has been only partially obtained. Moreover, the inequalities describing this polytope have coefficients in $\{-1, 0, 1\}$. This is not the case for the dominating set polytope in the node-variables dimension. It is known from [1] that for any integer $p$, there exists a facet defining inequality having coefficients in $\{1, \ldots, p\}$. We also show a decomposition theorem by means of 1-sums. Again this decomposition is much simpler with the extended formulation than with the node-variables formulation given in [2].

## 1 Introduction

Let $G = (V, A)$ be a directed graph, not necessarily connected, where each arc and each node has a cost (or a profit) associated with it. Consider the following version of the *uncapacitated facility location problem* (UFLP), where each location $v \in V$ has a weight $w(v)$ that corresponds to the revenue obtained by opening a facility at that location, minus the cost of building this facility. Each arc $(u, v) \in A$ has a weight $w(u, v)$ that represents the revenue obtained by assigning the customer $u$ to the opened facility at location $v$, minus the cost originated by this assignment. The goal is to select some nodes where facilities are opened and assign to them the non selected node in such a way that the overall profit is maximized. This version of the $UFLP$ is called the *prize-collecting uncapacitated facility location* (pc-UFLP). The following is a natural linear relaxation of the pc-UFLP.

$$\max \sum_{(u,v)\in A} w(u,v)x(u,v) + \sum_{v\in V} w(v)y(v) \tag{1}$$

$$\sum_{(u,v)\in A} x(u,v) + y(u) \leq 1 \quad \forall u \in V, \tag{2}$$

$$x(u,v) \leq y(v) \quad \forall (u,v) \in A, \tag{3}$$

$$x(u,v) \geq 0 \quad \forall (u,v) \in A, \tag{4}$$

$$y(v) \geq 0 \quad \forall v \in V. \tag{5}$$

Let $P(G)$ be the polytope defined by (2)-(4), and let $UFLP'(G)$ be the convex hull of $P(G) \cap \{0,1\}^{|V|+|A|}$. Clearly $UFLP'(G) \subseteq P(G)$.

Given a directed graph $G = (V, A)$, a subgraph induced by the nodes $v_1, \ldots, v_r$ of $G$ is called a *bidirected cycle* if the only arcs in this induced subgraph are $(v_i, v_{i+1})$ and $(v_{i+1}, v_i)$, for $i = 1, \ldots, r$, with $v_{r+1} = v_1$. We denote it by $BIC_r$. The first part of this paper is devoted to the study of

$UFLP'(G)$, when $G$ is a bidirected cycle. At first sight, the description of $UFLP'(BIC_n)$ seems easy because of the simple structure of $BIC_n$. We will show that we need to add the so-called *lifted g-odd cycle inequalities*, to complete its description. These inequalities define facets of $UFLP'(BIC_n)$, and are valid for $UFLP'(G)$ for any graph $G$. We also give a linear time algorithm to separate these inequalities.

To complete the description of $UFLP'(G)$ in a more general class of graphs, we consider the graphs $G = (V, A)$ that decompose by means of 1-sum. As a consequence we obtain a complete description of $UFLP'(G)$ when $G$ can be decomposed as 1-sums of bidirected cycles.

In the second part of this paper we discuss the consequences of these results when applied to the dominating set problem. More precisely, let $G = (V, E)$ be an undirected graph. A subset $D \subseteq V$ is called a *dominating set* if every node of $V \setminus D$ is adjacent to a node of $D$. The *minimum weight dominating set problem* (MWDSP) is to find a dominating set $D$ that minimizes $\sum_{v \in V} w(v)$, where $w(v)$ is a weight associated with each node $v \in V$. A natural linear relaxation of the MWDSP is defined by the linear program below

$$\min \sum_{v \in V} w(v)x(v) \tag{6}$$

$$x(N[v]) \geq 1 \quad \forall v \in V, \tag{7}$$

$$x(v) \geq 0 \quad \forall v \in V, \tag{8}$$

$$x(v) \leq 1 \quad \forall v \in V, \tag{9}$$

where $N[v]$ denotes the set of neighbors of $v$ including it. Define $DSP(G)$ to be the convex hull of the integer vectors satisfying (7)-(9).

The MWDSP is a special case of the set covering problem. It is NP-hard even when all the weights are equal to 1, this can be shown using a simple reduction from the vertex cover problem. A large literature is devoted to this case and many of its variants, for a deep understanding of the subject we refer to [3,4]. It has been shown that when the weights are all equal to 1, the MWDSP is solvable in many classes of graphs, a non-exhaustive list is cactus graphs [5], trees [5], series-parallel graphs [6], permutation graphs [7,8,9,10], cocomparability graphs [11], (see chapter 2 in [4] for more classes). For the weighted case of the MWDSP we have a short list of graphs where this problem can be solved in polynomial time, for threshold graphs [12], for cycles [13] and for strongly chordal graphs [14]. Little is known from the point of view of polyhedral approach and particularly few complete characterizations of the polytope associated with the MDWSP are known. For the case of strongly chordal graph Farber [14] gives a primal-dual algorithm to solve the MWDSP this shows that $DSP(G)$ is defined by (7)-(9). $DSP(G)$ has been described for threshold graphs [12]. And it has been, first, characterized for cycle graphs in [13] and later published in [1]. This result has also been established in [15] using a different approach. One can also use the results related to the set covering polytope [16,17,18,19,20], to cite a few, to establish new results for the MWDSP. The set covering polytope is the convex hull of $\{x \in \mathbb{R}^n : \ Ax \geq 1, \ x \in \{0,1\}^n\}$, where $A$ is an $m \times n$ matrix with 0,1 entries. For example, the polytope $DSP(G)$ when $G$ is a cycle with $n$ nodes coincide with the set covering polytope when $A$ is the $C_n^3$ circulant matrix. Recently in [21] a complete description of the set covering polytope is established when $A$ is the circulant matrix $C_{2k}^k$ or $C_{3k}^k$, $k \geq 3$.

We give an extended formulation via facility location to completely characterize the $DSP(G)$ when $G$ is a cactus. This description has been studied in the original dimension that is $\mathbb{R}^{|V|}$ in [13,1]. They developed several facet defining inequalities for this case, and showed that this polytope has a more complicated structure than the case when $G$ is a cycle. Even with the 1-sum composition developed in [2], the complete characterization of $DSP(G)$ in cactus graphs has not been found. The main difficulty reported in [13,1] is the description of the polytope when restricted to the auxiliary graphs obtained after the decomposition. In our work we show that with the extended formulation this task is easy and allows us to completely describe this polytope in a higher dimension. Moreover in [13,1], it has been shown that for any fixed integer $p$, there exist a cactus $G$ such that $DSP(G)$ has a facet defining inequality with coefficients $1, \dots, p$. In our description all the facets defining inequalities have coefficients in $\{0, -1, +1\}$.

This paper is organized as follows. In Section 2, we give some useful definitions and notations. Section 3 is devoted to the characterization of $UFLP'(G)$ when $G$ is a bidirected cycle. In Section 4, we show how the results of the previous sections apply to the dominating set polytope using a composition theorem. Finally, in Section 5 we present the algorithmic consequences of our approach. In particular, we devise the first polynomial time algorithm to solve the MWDSP in cacti. This is done via a linear time separation algorithm of the inequalities we introduced.

## 2 Definitions and notations

Recall that a bidirected cycle $BIC_r$ of a directed graph $G = (V, A)$ is a sequence of nodes $v_1, \dots, v_n$ in $V$ and arcs $(v_i, v_{i+1})$, $(v_{i+1}, v_i)$ in $A$, for $i = 1, \dots, n$, where $v_{n+1} = v_1$. The arcs of $BIC_n$ are denoted by $A(BIC_n)$. To simplify the notation, we will denote the nodes of $BIC_n$ by $1, \dots, n$, and the arcs by $(i, i+1)$ and $(i+1, i)$ for $i = 1, \dots, n$. When we use numbers $i + j$ or $i - j$, $i, j \in \{1, n\}$, the positive numbers are taken modulo $n$ and the negative ones are taken modulo $-n$. The number zero represents the node $n$. A *bidirected path* $P$ of the graph $BIC_n$ is an ordered sequence of consecutive nodes of $BIC_n$, where the arcs $(i, i+1)$ and $(i+1, i)$ of any two consecutive nodes $i$ and $i+1$ of $P$, are both considered in the path. Here $i+1$ is taken modulo $n$. The size of $P$ is the number of its nodes minus one. Given a directed graph $G = (V, A)$ its intersection graph denoted by $I(G)$ is obtained by associating a node for each arc of $A$. Two nodes are adjacent if the tail of one of the corresponding arcs coincides with the tail or the head of the other corresponding arc. It is easy to see that $I(BIC_n)$ consists of the following *circulant graph* $G_{2n} = (A(BIC_n), E)$, where $A = \{a_1, \dots, a_{2n}\}$ and the set of edges $E$ consists of the edges $\{a_i, a_{i+1}\}$ and $\{a_i, a_{i+2}\}$, for $i = 1, \dots, 2n$; the indices are taken modulo $2n$.

For a directed graph $D = (V, A)$, and $S \subseteq V$, we denote by $\delta^+(S)$ the set of arcs $(u, v) \in A$ with $u \in S$ and $v \in V \setminus S$. For a node $v \in V$ we write $\delta^+(v)$ instead of $\delta^+(\{v\})$. If there is a risk of confusion we use $\delta^+_G$.

Given an undirected graph $G = (V, E)$, a subset $S \subseteq V$ is called *stable* if there is no edge between any pair of nodes of $S$. The convex hull of the incidence vectors of the stable sets in $G$ is called the *stable set polytope* and is denoted by $SSP(G)$. When each node $v \in V$ has an associated weight $w(v)$, the *maximum weight stable set problem* (MWSSP) is to find a stable set $S \subseteq V$ maximizing $\sum_{v \in S} w(v)$. A set $K \subseteq V$ is called a *clique* if there is an edge between every pair of nodes in $K$.

For a ground set $U$ and a function $f$ from $U$ to $\mathbb{R}$, we use $f(S)$ to denote $f(S) = \sum_{a \in S} f(a)$, whenever $S \subseteq U$.

## 3 The characterization of $UFLP'(BIC_n)$

First we will give two families of valid inequalities for $UFLP'(G)$, when $G$ is any directed graph.

Let $G = (V, A)$ be any directed graph. Let $BIC_r$ a bidirected cycle included in $G$. The inequality below is called a *bidirected cycle inequality* and has been introduced in [22],

$$\sum_{a \in A(BIC_r)} x(a) \leq \left\lfloor \frac{2|r|}{3} \right\rfloor . \tag{10}$$

Now let us introduce the g-odd cycle inequalities. For any directed graph $G = (V, A)$, a simple cycle $C$ is an ordered sequence $v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$, where $v_0 = v_p$ and for $i = 0, \ldots, p-1$, $v_i$ and $a_i$ are distinct nodes and arcs, respectively. For $i = 0, \ldots, p-1$, the nodes $v_i$ and $v_{i+1}$ are the endnodes of $a_i$.

By setting $a_p = a_0$, we associate with $C$ three more sets as below.

- We denote by $\hat{C}$ the set of nodes $v_i$, such that $v_i$ is the head of $a_{i-1}$ and also the head of $a_i$, $1 \leq i \leq p$.
- We denote by $\dot{C}$ the set of nodes $v_i$, such that $v_i$ is the tail of $a_{i-1}$ and also the tail of $a_i$, $1 \leq i \leq p$.
- We denote by $\tilde{C}$ the set of nodes $v_i$, such that either $v_i$ is the head of $a_{i-1}$ and also the tail of $a_i$, or $v_i$ is the tail of $a_{i-1}$ and also the head of $a_i$, $1 \leq i \leq p$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle will be called *g-odd* (generalized odd) if $p + |\dot{C}|$ (or $|\dot{C}| + |\tilde{C}|$) is odd, otherwise it will be called *g-even*. A cycle $C$ with $\dot{C} = \hat{C} = \emptyset$ is a *directed* cycle. The set of arcs in $C$ is denoted by $A(C)$.

Let $C$ be a g-odd cycle. Now we define a set of arcs $\tilde{A}(C)$ as follows. For each node $v_i \in \dot{C}$ we have two cases. Let $v_{i-1}$ and $v_{i+1}$ be the two neighbors of $v_i$ in $C$.

- If $v_{i-1}$ and $v_{i+1}$ are in $\tilde{C}$, we pick arbitrarily one arc from $\{(v_{i-1}, v_i), (v_{i+1}, v_i)\}$ and add it to $\tilde{A}(C)$.
- If only one of the neighbors of $v_i$ is in $\tilde{C}$, say the node $v_j \in \{v_{i-1}, v_{i+1}\}$. We add $(v_j, v_i)$ to $\tilde{A}(C)$.

Once the *lifting set* $\tilde{A}(C)$ has been defined, a *lifted g-odd cycle inequality* has the form

$$\sum_{a \in A(C)} x(a) + \sum_{a \in \tilde{A}(C)} x(a) - \sum_{v \in \hat{C}} y(v) \leq \frac{|\tilde{C}| + |\hat{C}| - 1}{2}. \tag{11}$$

One can easily show that this is a Gomory-Chvátal cut of rank one. Notice that given a g-odd cycle $C$, we might have several lifting sets $\tilde{A}(C)$, therefore we might have several lifted g-odd cycle inequalities. Similar inequalities called lifted odd cycle inequalities have been studied in [24,25,26,27].

The main result of this section is the following theorem.

**Theorem 1.** *$UFLP'(BIC_n)$ is described by the constraints (2)-(5), the bidirected cycle inequality (10) with respect to $BIC_n$ and the lifted g-odd cycle inequalities (11).*

The remainder of this section is devoted to prove this theorem. It is easy to see that $UFLP'(G)$ is full dimensional for any graph $G$. Now assume that

$$\alpha x + \beta y \leq \rho, \tag{12}$$

is a valid inequality defining a facet of $UFLP'(BIC_n)$. Let $F_{\alpha,\beta} = \{(x,y) \in UFLP(G) \cap \{0,1\}^{|V|+|A|} : \alpha x + \beta y = \rho\}$. We will show that (12) is one of the inequalities (2)-(5), (10) or (11). We assume in this section that (12) is different from (2)-(5) and (10). We will recall this when needed. In the proof we will implicitly use the following remark.

*Remark 1.* There exist always a feasible 0-1 solution in $F_{\alpha,\beta}$ that satisfies inequalities (2)-(5) as a strict inequalities (not necessarily at the same time). Otherwise (12) is one of the inequalities (2)-(5).

Now we give a series of technical lemmas that will be used in the discussion that complete the proof in subsection 3.1. For a detailed proofs see [28].

**Lemma 1.** *We have $\alpha(u,v) \in \{0,1\}$ for each $(u,v) \in A(BIC_n)$ and $\beta(u) \in \{0,-1\}$ for each $u \in V(BIC_n)$.*

*Proof.* The main idea of the proof is a transformation to the stable set polytope. We add a slack variable to each inequality (2), then we eliminate the $y$'s variables using the equations obtained from (2) after the additions of the slack variables. It is not difficult to see that the convex hull of the 0-1 solutions in this new system is exactly the stable set polytope of a graph $H = (U,E)$. Each column corresponds to a node in $H$, and two nodes are adjacent if there is some inequality so that the two respective columns appear with non zero coefficients. We can observe that this graph is quasi-line. Using the results in [29], we show that the inequalities defining the stable set polytope in this new graph can have coefficients in $\{0,1,2\}$. And we know that any valid inequality of $UFLP'(G)$ can be obtained from a valid inequality of that stable set polytope by eliminating the slack variables using the equations obtained from (2). This yield to a valid inequality with coefficients in $\{0,1\}$ for the $x$'s variables and with coefficients in $\{0,-1\}$ for the $y$'s variables. $\square$

**Lemma 2.** *We cannot have $\alpha(u,v) = 1$ for all $(u,v) \in A(BIC_n)$ and $\beta(u) = -1$ for all $u \in V(BIC_n)$.*

*Proof.* Assume that $\alpha(u,v) = 1$ for all $(u,v) \in A(BIC_n)$ and $\beta(u) = -1$ for all $u \in V(BIC_n)$ . Notice that

$$\text{Max}_{(x,y)\in UFLP'(BIC_n)} \left\{ \sum_{(u,v)\in A(BIC_n)} x(u,v) - \sum_{u\in V(BIC_n)} y(u) \right\} = \left\lfloor \frac{n}{3} \right\rfloor . \tag{13}$$

In fact, notice that if $y(i) = 1$, to have a positive contribution to the objective, we need $x(i-1,i) = 1 = x(i+1,i)$. So we should have a configuration like in Figure 1. The maximum number of such configurations is $\lfloor \frac{n}{3} \rfloor$. It follows that

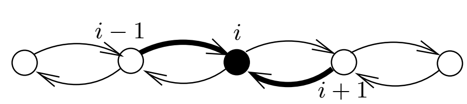


**Fig. 1.** The black node and bold arcs are variables with value 1, the other variables are zero, except the nodes in the extremities.

$$\left\lfloor \frac{n}{3} \right\rfloor \le \rho \tag{14}$$

We will study three cases. If $n = 3k$, there are only 3 feasible $0-1$ vectors that give the maximum of (13). If $n = 3k+1$, there are only $n$ feasible $0-1$ vectors that give this maximum of (13). Since we have a full dimensional polytope we need at least $3n$ vectors that satisfy the inequality as equation, to have a facet. Therefore, to finish the proof we need to study the case $n = 3k+2$ differently, since int his case we have $3n$ vectors that give the maximum of (13).

Let $n = 3k+2$. The nodes of $BIC_n$ are $1, \dots, 3k, 3k+1, 3k+2$.

We define a cycle $C$ as follows. The set of nodes in $\hat{C}$ are $i = 2+3l$, for $l = 0, \dots, k-1$ plus the node $3k+1$. For each node $i \in \hat{C}$ let $(i-1, i)$ and $(i+1, i)$ in $A(C)$. To complete the cycle we add the arcs $(3k+2, 1)$ and $(i, i+1)$ for $i = 3l$, $l = 1, \dots k$. It results that the nodes $i = (3k+2)+3l$ for $l = 0, \dots, k$, are in $\dot{C}$ and the nodes $i = (3k+2)+3l+1$ for $l = 0, \dots, k-1$, are in $\tilde{C}$. Hence $|\hat{C}| = |\dot{C}| = k+1$ and $|\tilde{C}| = k$, so $C$ is a $g$-odd cycle. Define a lifting set as follows: with $\tilde{A}(C) = \{(i+1, i) : i = (3k+2)+3l, l = 0, \dots, k-1\}$. We have the following lifted g-odd cycle inequality,

$$\sum_{a \in A(C)} x(a) + \sum_{a \in \tilde{A}(C)} x(a) - \sum_{v \in \hat{C}} y(v) \le \frac{|\hat{C}| + |\tilde{C}| - 1}{2} = k = \left\lfloor \frac{n}{3} \right\rfloor. \tag{15}$$

Inequalities (3) imply,

$$x(i, i-1) \le y(i-1) \text{ for each } i \in \hat{C}, \tag{16}$$
$$x(i, i+1) \le y(i+1) \text{ for each } i \in \hat{C}. \tag{17}$$

The sum of (15), (16) and (17) shows that (12) cannot define a facet of $UFLP'(BIC_n)$. □

The following three lemmas are easy to prove.

**Lemma 3.** *Let $i$ be a node of $BIC_n$ with $\beta(i) = -1$. Then $\alpha(i+1, i) = \alpha(i-1, i) = 1$.*

**Lemma 4.** *Let $i$ be a node of $BIC_n$ with $\beta(i) = -1$. If $\alpha(i, i-1) = \alpha(i-1, i) = 1$, then $\beta(i-1) = -1$.*

**Lemma 5.** *Let $i$ be a node of $BIC_n$ with $\beta(i) = -1$. If $\alpha(i, i-1) = 1$, then $\alpha(i, i+1) = 1$.*

Let us summarize the implications of Lemmas 2, 3, 4 and 5 by the following Lemma.

**Lemma 6.** *Let $i$ be a node of $BIC_n$ with $\beta(i) = -1$. Then the following assumptions hold*

*(a1)* $\alpha(i+1, i) = \alpha(i-1, i) = 1$, *and*
*(a2)* $\alpha(i, i-1) = \alpha(i, i+1) = 0$.

*Proof.* (a1) is obtained from Lemma 3. Now if we suppose that (a2) is not true, then Lemma 4 and Lemma 5 imply that $\alpha(u, v) = 1$ for each $(u, v) \in A(BIC_n)$ and $\beta(u) = -1$ for each $u \in V(BIC_n)$. But this contradicts Lemma 2. □

**Lemma 7.** *If $\alpha(i-1, i) = 1$ and $\beta(i) = 0$, then $\alpha(i, i+1) = 1$.*

*Proof.* There is a vector $x \in F_{\alpha,\beta}$ with $y(i-1) + x(i-1, i) + x(i-1, i-2) = 0$.

– If $y(i) = 1$, we set $x(i-1, i) = 1$ and violate the inequality; so $y(i) = 0$.
– If $x(i, i+1) = 0$, then we can set $y(i) = 1$ and proceed as before; so $x(i, i+1) = 1$.
– If $\alpha(i, i+1) = 0$, we set $x(i, i+1) = 0$ and proceed as before; so $\alpha(i, i+1) = 1$. □

**Lemma 8.** *Suppose that we are not dealing with the bidirected cycle inequality. If* $\alpha(i, i+1) = \alpha(i+1, i) = 1$ *then* $\alpha(i+2, i+1) = \alpha(i-1, i) = 0$.

*Proof.* Assume $i = 1$. The proof is based on the statements below.

– It follows from Lemma 6 that $\beta(1) = \beta(2) = 0$.
– It follows from Lemma 7 that $\alpha(2, 3) = \alpha(1, n) = 1$.
– Since this is not a bidirected cycle inequality, we assume that there is an index $k \geq 2$ such that:
  - $\beta(j) = \beta(j+1) = 0$, $\alpha(j, j+1) = \alpha(j+1, j) = 1$, for $1 \leq j \leq k$.
  - $\alpha(n, 1) = \alpha(k+2, k+1) = 0$.
  - $\alpha(1, n) = \alpha(k+1, k+2) = 1$.
– There is a vector $x \in F_{\alpha,\beta}$ with $y(k-1) + x(\delta^+(k-1)) = 0$. We modify $x$ as below to obtain a vector that violates the inequality.
  - If $y(k) = 1$ we just set $x(k-1, k) = 1$.
  - If $y(k) = 0$ and $x(k, k+1) = 0$, we set $y(k) = 1$ and proceed as above.
  - If $y(k) = 0$ and $x(k, k+1) = 1$, we set $y(k+1) = x(k, k+1) = x(k+2, k+1) = 0$, and $y(k) = x(k-1, k) = x(k+1, k) = 1$. □

**Lemma 9.** *If* $\alpha(i-1, i) = \alpha(i+1, i) = 1$*, then* $\beta(i) = -1$.

*Proof.* Suppose $\beta(i) = 0$. It follows from Lemma 7 that $\alpha(i, i-1) = \alpha(i, i+1) = 1$. This contradicts Lemma 8. □

**Lemma 10.** *We have at least one of the values* $\alpha(i, i+1)$ *or* $\alpha(i+1, i)$ *equal to 1, for each* $i = 1, \ldots, n$.

Now we can complete the proof of Theorem 1.

### 3.1 The proof of Theorem 1

Let $G_\alpha$ be the graph induced by the arcs $(i, j) \in A(BIC_n)$ with $\alpha(i, j) = 1$, we call this graph the *support graph* of (12). Recall that a bidirected path $P$ of a graph $G = (V, A)$ is a sequence of nodes $P = 1, 2, \ldots, k$ with $(i, i+1)$ and $(i+1, i)$ are both in $A$, for $i = 1, \ldots, k-1$. The size of $P$ is $k-1$. We say that $P$ is *maxima*l if we cannot extend it to a bidirected path from one of its endnodes.

Notice that by definition the support graph of any g-odd lifted cycle inequality satisfy the following three properties

– it contains a cycle as a subgraph,
– each maximal bidirected path is of size 1. Moreover, if $P = i, i+1$ is such a path, then $(i-1, i)$ and $(i+2, i+1)$ do not appear, and
– if $C$ is the lifted cycle and $i$ a node in $\dot{C}$, then the support graph must contain exactly one of the arcs $(i-1, i)$ or $(i+1, i)$ when both nodes $i-1$ and $i+1$ are in $\tilde{C}$, it contains none of the arcs if both of these nodes are in $\hat{C}$ and finally if, say $i+1$ is in $\tilde{C}$, we must have the arc $(i+1, i)$.

Let us see that these properties are satisfied by $G_\alpha$. Lemma 10 implies that $G_\alpha$ contains at least one cycle as a subgraph. Choose any such a cycle and call it $C$. Lemma 8 implies that each maximal bidirected path is of size one, and that for any such bidirected path $P = i, i+1$ the arcs $(i-1, i)$ and $(i+2, i+1)$ are not in $G_\alpha$. Again Lemma 10 implies that $(i, i-1)$ and $(i+1, i+2)$ belong to $G_\alpha$.

Let $i \in \dot{C}$, and let $i-1$ and $i+1$ be the neighbors of $i$ in $G_\alpha$. Notice that $G_\alpha$ must contain at most one of the arcs $(i-1, i)$ and $(i+1, i)$ since the size of maximal bidirected path is one.

If both $i-1$ and $i+1$ are in $\hat{C}$, then Lemma 9 implies that $\beta(i-1) = -1 = \beta(i+1)$, and using Lemma 6 we obtain that $\alpha(i-1, i) = 0 = \alpha(i+1, i)$. So in this case the arcs $(i-1, i)$ and $(i+1, i)$ are not in $G_\alpha$.

Assume that $i+1$ is in $\tilde{C}$ and that $G_\alpha$ contains none of the arcs $(i-1, i)$ or $(i+1, i)$, that is $\alpha(i-1, i) = \alpha(i+1, i) = 0$. By definition $\alpha(i, i-1) = \alpha(i, i+1) = 1$. Lemma 6 implies that $\beta(i) = 0$ and since $i+1$ is in $\tilde{C}$, we must have $\alpha(i+1, i+2) = 1$ and then again Lemma 6 implies that $\beta(i+1) = 0$. We can assume that there is a solution $(x, y) \in F_{\alpha,\beta}$ with $x(i+1, i) = 1$, otherwise (12) is the trivial inequality $x(i+1, i) \geq 0$. Now if we set $x(i, i+1)$ and $y(i+1)$ to 1; $x(i+1, i)$ and $y(i)$ to 0 and possibly $x(i-1, i)$ to 0, we obtain a feasible solution that violates (12). Therefore, we must have exactly one of the arcs $(i-1, i)$ or $(i+1, i)$ in $G_\alpha$. Moreover, if the node $i-1$ is in $\hat{C}$, Lemma 9 implies that $\beta(i-1) = -1$, and Lemma 6 implies that $\alpha(i-1, i) = 0$, so $(i-1, i)$ is not an arc of $G_\alpha$.

The above discussion shows that the support graph $G_\alpha$ coincides with the support graph of the lifted g-odd cycle inequality defined from $C$. Moreover, from Lemma 6, each node $i$ with $\beta(i) = -1$ must be in $\hat{C}$. And from Lemma 9, for each node $i \in \hat{C}$ we have $\beta(i) = -1$.

For a g-odd cycle inequality it is easy to find a 0-1 vector of $UFLP'(BIC_n)$ that satisfies it with equation. Then we have $\rho \geq (|\hat{C}| + |\tilde{C}| - 1)/2$.

Now the proof of Theorem 1 is complete.

## 4 Application to the dominating set polytope

Let $G = (V, E)$ an undirected connected graph. The graph $G$ is a *cactus* if each edge of $G$ is contained in at most one cycle of $G$. For example every tree is a cactus. The main result of this section is a complete description of the dominating set polytope $DSP(G)$ in $\mathbb{R}^{|V|+2|E|}$ when $G$ is a cactus. This description can be seen as an extended formulation of $DSP(G)$. We will show that with this extended formulation, to obtain the polytope associated with a cactus, it suffices to characterize the polytope associated with the maximal two-connected components.

Given an undirected graph $G = (V, E)$. We say that $G$ is a 1-sum of $G_1 = (V_1, E_1)$ and $G_1 = (V_1, E_1)$ if $|V_1 \cap V_2| = 1$, $V = V_1 \cup V_2$, $E = E_1 \cup E_2$.

Consider the following equalities obtained from (2).

$$\sum_{(u,v) \in A} x(u, v) + y(u) = 1 \quad \forall u \in V. \tag{18}$$

Define $UFLP(G)$ to be the convex hull of the feasible 0-1 vectors satisfying (18) and (3)-(5). This is the classical uncapacitated facility location polytope. Now given an undirected graph $G = (V, E)$, define the directed graph $\overleftrightarrow{G} = (V, A)$ that have the same node-set as $G$, and its arc-set $A$ is defined from $E$ by replacing each edge $uv \in E$ by two arcs $(u, v)$ and $(v, u)$.

**Lemma 11.** *For any undirected graph $G = (V, E)$, the projection of $UFLP(\overleftrightarrow{G})$ onto the $y$'s variables is exactly $DSP(G)$.*

*Proof.* We have to prove, $DSP(G) = \{y \mid \text{ there is a vector } x \text{ such that } (x, y) \in UFLP(\overleftrightarrow{G})\}$. First consider $\bar{y} \in DSP(G)$. We have $\bar{y} = \sum \alpha_i y^i$, $\sum \alpha_i = 1$, $\alpha \geq 0$, where $\{y^i\}$ are extreme points of $DSP(G)$. Consider now a particular vector $y^k$. Let $D^k = \{u \mid y^k(u) = 1\}$. For each $v \in V \setminus D^k$, there is at least one of its neighbors in $D^k$, $w_v$ say. We set $x^k(v, w_v) = 1$. We set $x^k(i, j) = 0$ for all other arcs $(i, j)$ in $\overleftrightarrow{G}$. Each vector $(x^k, y^k)$ is an extreme point of $UFLP(\overleftrightarrow{G})$. So $(\bar{x}, \bar{y}) = \sum \alpha_i (x^i, y^i)$ is a vector in $UFLP(\overleftrightarrow{G})$. Consider now $(\bar{x}, \bar{y}) \in UFLP(\overleftrightarrow{G})$. We have $(\bar{x}, \bar{y}) = \sum \alpha_i (x^i, y^i)$, $\sum \alpha_i = 1$, $\alpha \geq 0$, where each vector $(x^i, y^i)$ is an extreme point of $UFLP(\overleftrightarrow{G})$. Then each vector $y^i$ is the incidence vector of a dominating set $D^i$, therefore it is an extreme point of $DSP(G)$. Then $\bar{y} = \sum \alpha_i y^i$ is a vector in $DSP(G)$. □

**Theorem 2 ([30]).** *Let $D$ be a directed graph that is a 1-sum of $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, with $V_1 \cap V_2 = \{u\}$. Let $D'_1$ be the graph obtained from $D_1$ by replacing $u$ with $u'$, and $D'_2$ is obtained from $D_2$ by replacing $u$ with $u''$. Suppose that the system*

$$Az' \leq b \tag{19}$$

$$z'\left(\delta^+_{D'_1}(u')\right) + z'(u') \leq 1 \tag{20}$$

*describes $UFLP'(D'_1)$. Suppose that (19) contains the inequalities (2)-(5) except for (20). Similarly suppose that*

$$Cz'' \leq d \tag{21}$$

$$z''\left(\delta^+_{D'_2}(u'')\right) + z''(u'') \leq 1 \tag{22}$$

*describes $UFLP'(D'_2)$. Also (21) contains the inequalities (2)-(5) except for (22). Then the system below describes an integral polyhedron.*

$$Az' \leq b \tag{23}$$

$$Cz'' \leq d \tag{24}$$

$$z'\left(\delta^+_{D'_1}(u')\right) + z''\left(\delta^+_{D'_2}(u'')\right) + z'(u') \leq 1 \tag{25}$$

$$z'(u') = z''(u''). \tag{26}$$

Thus the theorem below follows from Theorem 1 and Theorem 2.

**Theorem 3.** *If $G$ is a cactus, then $UFLP'(\overleftrightarrow{G})$ is described by the constraints (2)-(5), the bidirected cycle inequalities (10), and the lifted g-odd cycle inequalities (11).*

$UFLP(\overleftrightarrow{G})$ is a face of $UFLP'(\overleftrightarrow{G})$. From Lemma 11, $DSP(G)$ is a projection of $UFLP(\overleftrightarrow{G})$. Therefore we have an extended formulation for $DSP(G)$.

## 5 Algorithmic consequences

In [1] the authors give the first polynomial algorithm to solve the minimum weighted dominating set problem (MWDSP) in a cycle. They showed that the separation of the inequalities defining the dominating set polytope in a cycle can be done in $O(n^2)$. Below we will show that the separation of our inequalities can be done in linear time. In [31], we gave a linear time combinatorial algorithm that solves the MWDSP, when the underlying graph is a cactus. Below we will give a cutting-plane polynomial time algorithm to solve pc-UFLP in the graph $\overleftrightarrow{G}$ when $G$ is a cactus. From Theorem 3 it suffices to develop a polynomial time algorithm to solve the separation problem associated with inequalities (10) and (11). Recall that $\overleftrightarrow{G}$ can be decomposed by means of 1-sum into bidirected cycles and bidirected paths of size one. The number of bidirected cycles is at most $\frac{n}{3}$, where $n$ is the number of nodes of $G$. It follows that one can easily introduce the bidirected cycle inequalities (10) in any linear program. Therefore we only need to solve the separation problem for the lifted g-odd inequalities (11) for each component of $\overleftrightarrow{G}$ that is a bidirected cycle.

**Separating lifted g-odd inequalities in a bidirected cycle** Given a vector $(x, y)$ we want to verify if there is a lifted g-odd cycle inequality (11) violated by $(x, y)$ if there is any.

**Theorem 4.** *The g-odd lifted cycle inequalities can be separated in linear time for bidirected cycles.*

*Proof.* A lifted g-odd cycle inequality (11) can also be written as

$$\sum_{a \in A(C)} (1 - 2x(a)) - \sum_{a \in \tilde{A}(C)} 2x(a) + \sum_{v \in \hat{C}} (2y(v) - 1) \geq 1. \tag{27}$$

Thus we look for a cycle that violates (27). For that we create a directed graph $D' = (V', A')$ as follows. For every arc $(i, i+1)$ and $(i+1, i)$ we create a node in $D'$. The arcs in $A'$ are as below.

- From $(i, i+1)$ to $(i+1, i+2)$ we create an arc with weight $1 - 2x(i+1, i+2)$ and label "odd."
- From $(i, i+1)$ to $(i+2, i+1)$ we create an arc with weight $2y(i+1) - 2x(i+2, i+1)$ and label "even."
- From $(i+1, i)$ to $(i+1, i+2)$ we create an arc with weight $1 - 2x(i+1, i+2)$ and label "odd."
- From $(i+1, i)$ to $(i+2, i+1)$ we create an arc with weight $1 - 2x(i+2, i+1)$ and label "odd."
- From $(i, i-1)$ to $(i+1, i+2)$ we create an arc with weight $2 - 2x(i, i+1) - 2x(i+1, i) - 2x(i+1, i+2)$, and label "even." This arc corresponds to the case when either $(i, i+1)$ or $(i+1, i)$ is in the lifting set $\tilde{A}(C)$.

Then we look for a minimum weight directed cycle with an odd number of odd arcs in $D'$. If the weight of such a cycle is less than one, we have found a violated inequality.

Now we give the details of how to find a minimum weight directed cycle with an odd number of odd arcs. We pick and index $i$, and remove the arcs entering $(i, i+1)$ and $(i+1, i)$. We add an extra node $s$ and connect it to $(i, i+1)$ and $(i+1, i)$ with even arcs of weight zero. For each node $v$ in $D'$ let $f_o(v)$ (resp. $f_e(v)$) be the weight of a shortest path from $s$ to $v$ having an odd (resp. even) number of odd arcs. We set $f_e(s) = 0$, $f_o(s) = f_o(v) = f_e(v) = \infty$ for every other node $v$ in $D'$. We call the labels of $s$ permanent and all others temporary. For each arc $(u, v)$ we denote by $w(u, v)$ its weight. Then for a node $v$ such that all its predecessors have permanent labels we update its labels as below.

$$f_o(v) = \min\Big\{ \min_u\{f_o(u) + w(u,v) : (u,v) \text{ is even}\}, \min_u\{f_e(u) + w(u,v) : (u,v) \text{ is odd}\}\Big\} \quad (28)$$

$$f_e(v) = \min\Big\{ \min_u\{f_o(u) + w(u,v) : (u,v) \text{ is odd}\}, \min_u\{f_e(u) + w(u,v) : (u,v) \text{ is even}\}\Big\}. \quad (29)$$

Then the labels of $v$ are called permanent, and we continue.

Once all labels are permanent, we use the arcs entering $(i, i+1)$ and $(i+1, i)$ to find a shortest directed cycle with an odd number of odd arcs and including either $(i, i+1)$ or $(i+1, i)$. Next we have to consider the case when neither $(i, i+1)$ nor $(i+1, i)$ is in the shortest cycle. This is when the arc from $(i, i-1)$ to $(i+1, i+2)$ is part of the shortest cycle. For that we repeat the same procedure with $i' = i+1$.

Since the indegree of each node in $D'$ is at most three, the labels in (28) and (29) are computed in constant time for each node. Therefore this is a linear time algorithm. □

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