RC25466 (WAT1405-019) May 8, 2014
Mathematics

IBM Research Report

Control Localization in Dynamical Systems Connected via a Weighted Tree

Ravindra Bapat
Indian Statistical Institute
New Delhi
110016 India

Chai Wah Wu
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598
USA

Control localization in dynamical systems connected via a weighted tree

Ravindra Bapat† and Chai Wah Wu⋆
†Indian Statistical Institute, New Delhi, 110016, INDIA
⋆IBM T. J. Watson Research Center, P. O. Box 218, Yorktown Heights, NY 10598, U.S.A
contact email: chaiwahwu@member.ams.org

Abstract

We consider the problem of localization of control in dynamical systems coupled via a weighted tree, when only a single system receives control. We abstract this problem into a study of eigenvalues of a perturbed Laplacian matrix. We show that this eigenvalue problem has a complete solution for arbitrarily large control by showing that the best and the worst places to apply control must necessarily be a characteristic vertex and a pendant vertex, respectively. Some partial results are proved in the case of finite control. In particular, we show that a local maximum in localizing the best place for control is also a global maximum. We conjecture in the finite control case that the best place to apply control must also necessarily be a characteristic vertex and present evidence from numerical experiments to support this conjecture.

I. Introduction

There have been many studies on control of networks of dynamical systems where control is applied to a subset of systems [1], [2]. Of particular interest is the relationship between the locations where control is applied and the control effectiveness. In this paper, we consider state equations of the form:

$$\frac{dx_i}{dt} = f(x_i,t) - \alpha\left(\sum_j L_{ij}D(t)x_j + c_iD(t)\left(x_i - u(t)\right)\right) \tag{I.1}$$

where $u(t)$ is the desired target trajectory and c_i are scalar control strengths with $c_i > 0$ if control is applied to the i-th system and $c_i = 0$ otherwise. Assume that $u(t)$ is a trajectory of the individual (unforced) dynamical system in the network, i.e.,

$$\frac{du(t)}{dt} = f(u(t),t) \tag{I.2}$$

From L we can create the augmented matrix $\tilde{L}$ by adding a row and column:

$$\tilde{L} = \begin{pmatrix} L+C & -c \\ 0 & 0 \end{pmatrix}$$

where C is a diagonal matrix with c_i on the diagonal and c is the vector of c_i's.

Using $u(t) = x_{n+1}(t)$ (see also [3]) Eq. (I.1) can be rewritten as

$$\frac{dx_i}{dt} = f(x_i,t) - \alpha\left(\sum_j \tilde{L}_{ij}D(t)x_j\right) \tag{I.3}$$

Definition 1: Let B be an irreducible square matrix B with nonpositive off-diagonal elements. Decompose B uniquely as $B = L + U$, where L is a zero row sum matrix and U is a diagonal matrix. Let w be the unique positive vector such that $w^TL = 0$ and $\max_v w_v = 1$. Let $W = \mathrm{diag}(w)$. Then $\beta(B) = \min_{x\neq 0} \frac{x^TWBx}{x^TWx}$.

The vector w in Definition 1 exists by Theorem 2 (see below) and the fact that 0 is the smallest eigenvalue of L and has multiplicity 1. Consider the following result describing a sufficient condition under which the state trajectories x_i follow the target trajectory $u(t)$.

Theorem 1 ([5]): If the following conditions are satisfied:

1) L is a zero row sums matrix with nonpositive off-diagonal elements;
2) $\tilde{L}$ has a Frobenius normal form:

$$\tilde{L} = Q\begin{pmatrix} B_1 & B_{12} & \cdots & B_{1q} \\ & B_2 & \cdots & B_{2q} \\ & & \ddots & \vdots \\ & & & B_q \end{pmatrix}Q^T \tag{I.4}$$

for some permutation matrix Q and B_i are square irreducible matrices with $B_q = 0$.

3) $f(x,t) - D(t)x$ satisfies $(y-z)^T V(f(y,t) - f(z,t) - D(t)(y-z)) \leq -\mu \|y-z\|^2$ for some $\mu > 0$ and all y, z, t and some symmetric positive semidefinite matrix V.
4) $VD(t)$ is symmetric positive semidefinite for all t;
5) $\beta_{\min}(\tilde{L}) \geq \frac{1}{\alpha}$ where $\beta_{\min}(\tilde{L})$ is defined as $\min_{i<q} \beta(B_i)$.

Then $x_i(t) - u(t) \to 0$ as $t \to \infty$ for all i.

Theorem 1 indicates that $\beta_{\min}(\tilde{L})$ of the augmented matrix $\tilde{L}$ is important to the ability to synchronize. Of particular interest are the locations of the nonzero entries in the vector c that maximizes or minimizes $\beta_{\min}(\tilde{L})$. Even though Theorem 1 is a sufficient condition (it can be necessary as well if f is linear and time-invariant) we will use $\beta_{\min}$ as a proxy of how well the network is amenable to control. The purpose of this paper is to analyze the case where the underlying network is an undirected weighted tree and find out where on the network to apply control to maximize or minimize the control's effectiveness.

II. Mathematical preliminaries

A matrix $A = \{a_{ij}\}$ is *nonnegative* (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all i, j. The spectral radius of a matrix A, denoted $\rho(A)$, is the maximum of the absolute values of all its eigenvalues. We will use the following fundamental results from Perron-Frobenius theory of nonnegative matrices [4].

Theorem 2: A square nonnegative matrix A of order n has a real nonnegative eigenvalue equal to the spectral radius of A with a corresponding nonnegative eigenvector. If in addition the matrix A is irreducible, then there exists a real positive eigenvalue of multiplicity 1 equal to the spectral radius of A with a corresponding positive eigenvector.

Lemma 1:

1) Let A and B be nonnegative matrices such that $A \geq B$ (entrywise). Then $\rho(A) \geq \rho(B)$. The inequality is strict if A is irreducible and $A \neq B$.
2) Let A be a symmetric $n \times n$ matrix such that A is positive definite and the off-diagonal elements of A are nonpositive. Then A^{-1} is a nonnegative matrix. If A is irreducible, then A is positive.

Definition 2: Let A be an $m \times n$ matrix. The Moore-Penrose inverse of A, denoted $A^\dagger$, is the unique $n \times m$ matrix satisfying the equations

- $AA^\dagger A = A$,
- $A^\dagger A A^\dagger = A^\dagger$,
- $(AA^\dagger)^* = AA^\dagger$,
- $(A^\dagger A)^* = A^\dagger A$.

For basic properties of the Moore-Penrose inverse we refer the reader to [8]. We use the Moore-Penrose inverse merely as a notational device. In particular, we will use the following simple observation.

Lemma 2: Let A be an $n \times n$ matrix and suppose

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where B is square and nonsingular. Then

$$A^\dagger = \begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

A weighted graph is a simple graph such that each edge is assigned a *positive* weight. The weight of the edge e is denoted $w(e) > 0$. Let G be a weighted graph with $V(G) = \{1, \ldots, n\}$. The Laplacian matrix L of G is an $n \times n$ matrix defined as follows: for $i \neq j$, the (i,j)-element ℓ_{ij} of L is 0 if vertices i and j are not adjacent, and it equals $-w(e)$ if i and j are adjacent and $w(e)$ is the weight of the edge $e = \{i, j\}$. For $i = 1, \ldots, n$; the (i,i)-element of L equals the sum of the weights of the edges incident to vertex i. The Laplacian matrix is a symmetric, positive semidefinite matrix. It is singular and has rank $n-1$ if the graph is connected. If A is a symmetric $n \times n$ matrix, then $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$ will denote the eigenvalues of A.

When L is the Laplacian matrix of a weighted undirected tree, L is symmetric and, following the notation of Section I, $\beta_{\min}(\tilde{L}) = \lambda_1(L + C)$. Denote P as the set of indices where $c_i \neq 0$. Where should the nonzero entry of C be located such that $\lambda_1(L + C)$ is maximized or minimized? Consider the limiting case where the nonzero entry c_i of C approaches infinity: $\kappa = \lim_{c_i \to \infty} \lambda_1(L + C)$. Let $\eta_{\max}$ be the value of κ that is maximized over all possible location for the nonzero c_i in P.

Lemma 3 ([1]): If the number of nonzero c_i in C is equal to $p < n$, then

$$\lambda_1(L + C) \leq \lambda_{p+1}(L)$$

This result is a consequence of the Weyl Eigenvalue Interlacing Theorem.

Lemma 4 ([1]): $\kappa = \lambda_1(L_P)$ where L_P is the matrix obtained from L by deleting the columns and rows corresponding to nonzero c_i's.

Since L is singular, $\lambda_1(L) = 0$. The eigenvalue $\lambda_2(L)$ is known as the algebraic connectivity of the graph and it is positive if and only if the graph is connected. This terminology is due to Fiedler [6], who proved some fundamental properties of an eigenvector corresponding to the algebraic connectivity. In particular, for trees, we state the following result.

Theorem 3 ([6]): Let y be the eigenvector corresponding to $\lambda_2(L)$ of the Laplacian matrix of a weighted tree T (also called the *Fiedler* vector). Then there are two possibilities:

1) There exists a vertex i such that $y_i = 0$. In this case the subgraph of T induced by the set of vertices corresponding to the zero coordinates of y is connected. Moreover, there is a unique vertex (called the characteristic vertex) k such that $y_k = 0$, and k is adjacent to a vertex m with $y_m \neq 0$. Such a tree is called a Type I tree.
2) No entry of y is zero. Then there exists a unique edge $e = \{i, j\}$ where $y_i y_j < 0$. In this case $T \setminus \{e\}$ consists of two subtrees such that y_i are of the same sign within each subtree. The edge $e = \{i, j\}$ is called an characteristic edge of T and i and j are called characteristic vertices of T. Such a tree is called a Type II tree.

III. Arbitrarily large control where $c \longrightarrow \infty$

We will focus on the case where there is only one nonzero entry in C, i.e. $|P| = 1$ as this case is somewhat tractable[1]. Lemma 3 shows that $\lambda_1(L + C) \leq \lambda_2(L)$. First we introduce some notation. By abuse of notation, we set $0 < c \in \mathbb{R}$ as the sole nonzero element in the diagonal matrix C. In this section we study κ, i.e. the case when $c \to \infty$. In Section IV the case of finite c will be considered.

Definition 3: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$, and let L be the Laplacian matrix of T. For $i = 1, \ldots, n$; we set

(i) L_i to be the matrix obtained by deleting row and column i from L
(ii) $\hat{L}_i$ to be the matrix obtained by replacing the elements in row and column i by zeros in L
(iii) $\tilde{L}_i(c)$ to be the matrix obtained by adding $c > 0$ to the i-th diagonal element in L.

Note that L_i is a principal submatrix of $\hat{L}_i$. We continue to index the rows and the columns of L_i by $\{1, \ldots, n\} \setminus \{i\}$. Lemma 4 implies that

$$\lim_{c \to \infty} \lambda_1(\tilde{L}_i(c)) = \lambda_1(L_i).$$

We state the following result which will be used.

Theorem 4 ([7]): Consider a weighted tree with Laplacian matrix L and let $i \in \{1, \ldots, n\}$. Then the (k, j)-entry of L_i^{-1} is equal to $\sum_{e \in P_{k,j}} \frac{1}{w(e)}$ where $P_{k,j}$ is the set of edges of the tree which are on both the path from vertex k to vertex i and on the path from vertex j to vertex i.

The inverse of L_i is described in Theorem 4 and hence by Lemma 2, we have a description of $\hat{L}_i^\dagger$, given as follows.

Theorem 5: Consider a weighted tree with Laplacian matrix L and let $i \in \{1, \ldots, n\}$. Then for $k \neq i, j \neq i$, the (k, j)-entry of $\hat{L}_i^\dagger$ is equal to $\sum_{e \in P_{k,j}} \frac{1}{w(e)}$ where $P_{k,j}$ is the set of edges of the tree which are on both the path from vertex k to vertex i and on the path from vertex j to vertex i. If $k = i$, or if $j = i$, then the (k, j)-entry of $\hat{L}_i^\dagger$ is zero.

Let G be a graph and let v be a vertex of G. By $G \setminus v$ we mean the graph obtained by deleting v and all edges incident with v from G. If e is an edge of G, then $G \setminus e$ will denote the graph obtained by deleting e from G.

Lemma 5: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and let L be the Laplacian matrix of T. Let $e = \{i, j\}$ be an edge of T. Let $T^{(i)}$ and $T^{(j)}$ be the components of $T \setminus e$, containing i and j respectively. Then the matrix $X = \hat{L}_i^\dagger - \hat{L}_j^\dagger$ is given as follows:

$$x_{uv} = \begin{cases} -\frac{1}{w(e)}, & \text{if } u, v \in T^{(i)} \\ \frac{1}{w(e)}, & \text{if } u, v \in T^{(j)} \\ 0, & \text{otherwise.} \end{cases}$$

Proof: The result follows by an application of Theorem 4 and Lemma 2. □

For a tree with a vertex k, a branch of T at k is defined to be one of the connected components of $T \setminus k$. Thus the branches at k are in one-to-one correspondence with the edges incident with k. By Theorem 4, the (i, j)-entry of L_k^{-1} is positive if and only if i and j belong to the same branch of T at k. Thus, L_k^{-1} is similar via a permutation to a block diagonal matrix, where the number of diagonal blocks is the degree of vertex k. Furthermore, each such block is a positive matrix by Lemma 1. A branch at k is called a *Perron branch* if the corresponding block in L_k^{-1} has the maximum spectral radius among all the blocks, or equivalently, if the spectral radius of the corresponding block equals $\rho(L_k^{-1})$. We now state the following result.

Lemma 6 ([7]): Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and suppose that m is not a characteristic vertex of T. Then the unique Perron branch of T at m is the branch which contains the characteristic vertex (or vertices) of T.

Let A be an $n \times n$ matrix and let $\phi \neq S \subset \{1, \ldots, n\}$. We set $A[S, S]$ to be the principal submatrix of A whose rows and columns are indexed by S.

[1]For the case $|P| > 1$, some results can be found for special graphs in [2].

Lemma 7: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and let L be the Laplacian matrix of T. Let $e = \{i, j\}$ be an edge of T. Suppose that i is not a characteristic vertex of T and that the path from i to a characteristic vertex contains j. Then $\lambda_1(L_i) < \lambda_1(L_j)$.

Proof: Let $T^{(i)}$ and $T^{(j)}$ be the components of $T \setminus e$ containing i and j respectively. Let $S = V(T^{(j)})$. By Lemma 6, $\rho(L_i^{-1}) = \rho(\hat{L}_i^\dagger) = \rho(\hat{L}_i^\dagger[S,S])$ and $\rho(L_j^{-1}) = \rho(\hat{L}_j^\dagger) = \rho(\hat{L}_j^\dagger[S,S])$. By Lemma 5, $\hat{L}_i^\dagger[S,S] - \hat{L}_j^\dagger[S,S] > 0$ and hence by Lemma 1,(i), $\rho(\hat{L}_i^\dagger[S,S]) > \rho(\hat{L}_j^\dagger[S,S])$. It follows that $\rho(L_i^{-1}) > \rho(L_j^{-1})$, and therefore $\lambda_1(L_i) < \lambda_1(L_j)$. □

The following is our main result in this section, which readily follows from Lemma 7.

Theorem 6: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$, let L be the Laplacian matrix of T, and let $i \in \{1, \ldots, n\}$. Then $\lambda_1(L_i)$ decreases along any path which starts at a characteristic vertex, and does not include the other characteristic vertex (if there is one). In particular, $\lambda_1(L_i)$ is maximized only when i is a characteristic vertex and is minimized only when i is a pendant vertex.

IV. The case when c is finite

Let T be a weighted tree with vertex set $\{1, \ldots, n\}$, and let L be the Laplacian matrix of T. In this Section we consider the problem of maximizing and minimizing $\lambda_1(\tilde{L}_i(c)), i \in \{1, \ldots, n\}$ when c is finite and prove a result analogous to Theorem 6. Note that $\frac{1}{c}L$ is the Laplacian matrix of the weighted tree obtained by replacing each weight $w(e)$ by $w(e)/c$. Without loss of generality we assume that $c = 1$ and set $\tilde{L}_i = \tilde{L}_i(1)$. Then $\tilde{L}_i$ is a positive definite matrix and hence is nonsingular. Let J be the matrix of appropriate size with each element equal to 1.

Theorem 7 (Matrix-Tree theorem for weighted graphs [9]): Let L be the Laplacian matrix of a weighted graph. For a spanning tree T, let $\omega(T)$ be the product of all its weights. Any cofactor of L is equal to $\sum_{T \in H} \omega(T)$ where H is the set of all spanning trees.

Lemma 8: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and let L be the Laplacian matrix of T. Let $i \in \{1, \ldots, n\}$. Then $\tilde{L}_i^{-1} = \hat{L}_i^\dagger + J$.

Proof: Let ω be the product of all the edge weights. By Theorem 7, any cofactor in L equals ω. Since $\det(L) = 0$, this fact, and the multilinear property of the determinant, show that $\det \tilde{L}_i = \omega$. By the cofactor formula for the inverse, the (j,k)-element of $\tilde{L}_i^{-1}$ is given by the cofactor of the (j,k)-element in $\tilde{L}_i$, divided by ω. If $j \neq i, k \neq i$, then again by the multilinearity of the determinant, the cofactor of the (j,k)-element in $\tilde{L}_i$ equals the cofactor of the (j,k)-element in $\hat{L}_i$, plus ω, which is the (j,k)-element of $\omega(\hat{L}_i^\dagger + J)$. If $j = i$ or $k = i$, then the cofactor of the (j,k)-element in $\tilde{L}_i$ equals ω, while the (j,k)-element in $\hat{L}_i^\dagger$ is zero. Thus the (j,k)-element in $\tilde{L}_i^{-1}$ is equal to the (j,k)-element of $\hat{L}_i^\dagger + 1$ for all j,k. This completes the proof. □

The following easy consequence of Lemma 8 will be used.

Lemma 9: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and let L be the Laplacian matrix of T. Let $i \in \{1, \ldots, n\}$. Then $\tilde{L}_i(c)^{-1} = \hat{L}_i^\dagger + \frac{1}{c}J$.

Proof: Let $\tilde{M}_i(1) = \frac{1}{c}\tilde{L}_i(c)$. By Lemma 8 $\tilde{M}_i^{-1} = \hat{M}_i^\dagger + J$. $\tilde{L}_i^{-1}(c) = \frac{1}{c}\tilde{M}_i^{-1} = \frac{1}{c}\hat{M}_i^\dagger + \frac{1}{c}J = \hat{L}_i^\dagger + \frac{1}{c}J$. □

Lemma 10: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and let L be the Laplacian matrix of T. Let $e = \{i,j\}, f = \{j,k\}$ be distinct edges of T sharing a common vertex j. Then for any $c > 0$, $w(e)\tilde{L}_i(c)^{-1} + w(f)\tilde{L}_k(c)^{-1} - (w(e) + w(f))\tilde{L}_j(c)^{-1}$ is a nonnegative, nonzero matrix.

Proof: By Lemma 9,

$$\begin{aligned} & w(e)\tilde{L}_i(c)^{-1} + w(f)\tilde{L}_k(c)^{-1} - (w(e)+w(f))\tilde{L}_j(c)^{-1} \\ = \; & w(e)(\hat{L}_i^\dagger + \frac{1}{c}J) + w(f)(\hat{L}_k^\dagger + \frac{1}{c}J) - (w(e)+w(f))(\hat{L}_j^\dagger + \frac{1}{c}J) \\ = \; & w(e)\hat{L}_i^\dagger + w(f)\hat{L}_k^\dagger - (w(e)+w(f))\hat{L}_j^\dagger. \end{aligned}$$

Let $T^{(i)}, T^{(j)}$ and $T^{(k)}$ be the components of $T \setminus \{e, f\}$ containing i, j and k respectively. Let $X = w(e)\hat{L}_i^\dagger + w(f)\hat{L}_k^\dagger - (w(e)+w(f))\hat{L}_j^\dagger$. By an application of Theorem 4 and Lemma 2 we see that

$$x_{uv} = \begin{cases} 2, & \text{if } u, v \in T^{(j)} \\ 1, & \text{if } u \in T^{(i)}, v \in T^{(j)} \text{ or } u \in T^{(j)}, v \in T^{(i)} \\ 1, & \text{if } u \in T^{(k)}, v \in T^{(j)} \text{ or } u \in T^{(j)}, v \in T^{(k)} \\ 0, & \text{otherwise.} \end{cases}$$

Hence X is a nonnegative, nonzero matrix, completing the proof. □

Lemma 11: Let T be a weighted tree with vertex set $\{1, \ldots, n\}$ and let L be the Laplacian matrix of T. Let $e = \{i,j\}, f = \{j,k\}$ be distinct edges of T. Then for any $c > 0$,

$$w(e)\left(\frac{1}{\lambda_1(\tilde{L}_i(c))} - \frac{1}{\lambda_1(\tilde{L}_j(c))}\right) > w(f)\left(\frac{1}{\lambda_1(\tilde{L}_j(c))} - \frac{1}{\lambda_1(\tilde{L}_k(c))}\right).$$

Proof: By Lemma 10, $w(e)\tilde{L}_i(c)^{-1} + w(f)\tilde{L}_k(c)^{-1} - (w(e)+w(f))\tilde{L}_j(c)^{-1}$ is a nonnegative, nonzero matrix. By Lemma 9, $\tilde{L}_i(c)^{-1}, \tilde{L}_k(c)^{-1}$ are positive and hence by Lemma 1,

$$\rho((w(e)+w(f))\tilde{L}_j(c)^{-1}) < \rho(w(e)\tilde{L}_i(c)^{-1} + w(f)\tilde{L}_k(c)^{-1}). \tag{IV.1}$$

Note that $\tilde{L}_i(c)^{-1}, \tilde{L}_k(c)^{-1}$ are symmetric positive definite and hence by the Rayleigh-Ritz variational principle for the maximal eigenvalue,

$$\rho(w(e)\tilde{L}_i(c)^{-1} + w(f)\tilde{L}_k(c)^{-1}) \leq \rho(w(e)\tilde{L}_i(c)^{-1}) + \rho(w(f)\tilde{L}_k(c)^{-1}). \tag{IV.2}$$

It follows from (IV.1),(IV.2) that

$$(w(e)+w(f))\rho(\tilde{L}_j(c)^{-1}) < w(e)\rho(\tilde{L}_i(c)^{-1}) + w(f)\rho(\tilde{L}_k(c)^{-1}). \tag{IV.3}$$

From (IV.3), we get

$$w(e)(\rho(\tilde{L}_i(c)^{-1}) - \rho(\tilde{L}_j(c)^{-1})) + w(f)(\rho(\tilde{L}_k(c)^{-1}) - \rho(\tilde{L}_j(c)^{-1})) > 0. \tag{IV.4}$$

Since $\rho(\tilde{L}_j(c)^{-1}) = 1/\lambda_1(\tilde{L}_j(c)), \rho(\tilde{L}_i(c)^{-1}) = 1/\lambda_1(\tilde{L}_i(c))$ and $\rho(\tilde{L}_k(c)^{-1}) = 1/\lambda_1(\tilde{L}_k(c))$, the result follows from (IV.4). □

Lemma 12: Let T be a weighted tree with vertex set $\{1,\dots,n\}$ and let L be the Laplacian matrix of T. Let $e=\{i,j\}, f=\{j,k\}$ be distinct edges of T. Let $c>0$ be fixed. If $\lambda_1(\tilde{L}_i(c)) \geq \lambda_1(\tilde{L}_j(c))$, then $\lambda_1(\tilde{L}_j(c)) > \lambda_1(\tilde{L}_k(c))$.

Proof: The result follows from Lemma 11. □

The following is our main result in this section.

Theorem 8: Let T be a weighted tree with vertex set $\{1,\dots,n\}$ and let L be the Laplacian matrix of T. Let $c>0$ be fixed. Then the following assertions hold:

(i) If $\lambda_1(\tilde{L}_i(c))$ is maximized over $i \in \{1,\dots,n\}$ at $i=m$, then $\lambda_1(\tilde{L}_i(c))$ decreases along any path which starts at m. Furthermore, $\lambda_1(\tilde{L}_m(c)) > \lambda_1(\tilde{L}_i(c))$ if m is not adjacent to i.

(ii) There is a pendant vertex k such that $\lambda_1(\tilde{L}_i(c))$ is minimized over $i \in \{1,\dots,n\}$ at $i=k$.

(iii) The maximum of $\lambda_1(\tilde{L}_i(c))$ over $i \in \{1,\dots,n\}$ occurs either at exactly one vertex, or at two vertices which must be adjacent.

Proof: Assertion (i) follows easily from Lemma 12, while (ii) follows from (i). To prove (iii), let us suppose that $\lambda_1(\tilde{L}_i(c))$ is maximized at vertices p and q which are not adjacent. Let $p = v_1 \sim v_2 \sim \cdots \sim v_\ell = q$ be the path from p to q. A repeated application of Lemma 12 shows that $\lambda_1(\tilde{L}_p(c)) > \lambda_1(\tilde{L}_q(c))$ and $\lambda_1(\tilde{L}_q(c)) > \lambda_1(\tilde{L}_p(c))$, which is a contradiction. Hence (iii) is proved. □

Theorem 8 shows that $\lambda_1(\tilde{L}_i(c))$ is maximized at an interior (non-pendant) vertex. In addition, a vertex maximizing $\lambda_1(\tilde{L}_i(c))$ locally is also maximizing it globally. This suggests the following parallel algorithm of applying control. Each vertex checks whether applying the control to one of its neighbors rather than itself results in a decreasing λ_1. If so, the vertex is the place that maximizes λ_1.

Numerical experiments suggest that for any $c>0$, $\lambda_1(\tilde{L}_i(c))$ is maximized at a characteristic vertex. This can be proved for trees with some symmetry using Theorem 8, but the proof in general is elusive. For example if P_n is the path with n vertices, with all the edge weights equal to 1, then it follows from Theorem 8 that $\lambda_1(\tilde{L}_i(c))$ is maximized at the vertex $\frac{n+1}{2}$, if n is odd, and at the vertices $\frac{n}{2}, \frac{n}{2}+1$ if n is even, i.e. in these cases λ_i is maximized when the nonzero c is at any characteristic vertex.

This leads us to formulate the following conjecture:

Conjecture 1: For diagonal matrix C with $|P|=1$ and weighted trees with Laplacian matrix L, $\lambda_1(L+C)$ is maximized when $c_i \neq 0$ occurs on a characteristic vertex.

For unweighted trees of 9 vertices or less and $c=1$, we find that $\lambda_{\min}(L+C)$ is maximized at a characteristic vertex, and minimized at a pendant vertex. In addition, for Type II trees of 4 vertices or less, $\lambda_{\min}(L+C)$ is maximized at both characteristic vertices. For Type II trees of 5 vertices or more, there are example when $\lambda_{\min}(L+C)$ is maximized at one characteristic vertex, but not at the other. In fact, all the trees of 5 vertices except for the star graph (which is a type I tree) are of this nature.

V. Acknowledgments

The first author acknowledges support from the JC Bose Fellowship, Department of Science and Technology, Government of India.

REFERENCES

[1] C. W. Wu, "On control of networks of dynamical systems," in *Proceedings of IEEE International Symposium on Circuits and Systems*, 2010, pp. 3785–3788.

[2] ——, "Control of Networks of Coupled Dynamical Systems," in *Consensus and Synchronization in Complex Networks*, ser. Understanding Complex Systems. Springer, 2013, pp. 23–50.

[3] F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen, "Controllability of complex networks via pinning," *Physical Review E*, vol. 75, p. 046103, 2007.

[4] H. Minc, *Nonnegative Matrices*. New York: John Wiley & Sons, 1988.

[5] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, pp. 1057–1064, 2005.

[6] M. Fiedler, "A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory," *Czechoslovak Mathematical Journal*, vol. 25, no. 100, pp. 619–633, 1975.

[7] S. Kirkland, M. Neumann, and B. L. Shader, "Characteristic vertices of weighted trees via perron values," *Linear and Multilinear Algebra*, vol. 40, no. 4, pp. 311–325, 2008.

[8] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*. Dover Publications, 1991.

[9] W. T. Tutte, *Graph Theory*. Addison-Wesley, 1984.