RC25531 (WAT1504-032) April 15, 2015
Mathematics

IBM Research Report

Meertens Number and Its Variations

Chai Wah Wu
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598
USA

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Chai Wah Wu
IBM T. J. Watson Research Center
P. O. Box 218, Yorktown Heights, NY 10598, USA

April 11, 2015

Abstract

Meertens numbers are numbers introduced in [1] that are invariant under a Gödel-like encoding. In base 10, the only known Meertens number is 81312000. We look at some properties of Meertens numbers and consider variations of the concept.

1 Introduction

Kurt Gödel in his celebrated work on mathematical logic [2] uses an injective map from the set of finite sequences of symbols to the set of natural numbers in order to describe statements in logic as natural numbers and relating properties of mathematical proofs with properties of natural numbers. This approach is subsequently used by Alan Turing to define the notion of computable numbers [5]. The basic Gödel encoding is as follows: each symbol in the alphabet is mapped to a positive number. Thus a finite sequence of symbols $s_1 \cdots s_n$ is mapped to a sequence of positive numbers $m_1, \cdots, m_n$. This is then mapped to a natural number $G = \prod_{i=1}^n p_i^{m_i}$, where p_i is the i-th prime.

In [1], Richard Bird dedicated the number 81312000 to his friend Lambert Meertens and called it a Meertens number. He constructed this number using a mapping similar to the Gödel encoding.

Definition. Given a decimal representation $d_1 \cdots d_n$ of the number m, if $m = \prod_{i=1}^n p_i^{d_i}$ then m is called a *Meertens number*.

The only Meertens number known to date is $81312000 = 2^8 3^1 5^3 7^1 11^2$ [1]. David Applegate has conducted the search up to 10^{29} (see `https://oeis.org/A246532`). Note that the function $M(m) = \prod_{i=1}^n p_i^{d_i}$ is similar to the Gödel encoding function. However, unlike the Gödel encoding, this function is not injective. In particular, $M(10^k) = 2$ for all $k \geq 0$. Since $d_1 \neq 0$, it is clear that a Meertens number must necessarily be even.

2 Meertens number in other bases

As noted in [1], the concept of a Meertens number can be defined in other number bases as well, i.e. m is a Meertens number in base b if m satisfies $m = \prod_{i=1}^n p_i^{d_i} = \sum_{i=1}^n d_i b^{n-i}$ for some nonnegative integers $d_i < b$ with $d_1 > 0$. Table 2 lists some Meertens numbers found in various number bases.

The number 82944 is interesting as it shares the first 4 digits with the base 8294. 82944 in base 8294 is $A4$ (where we borrow from hexadecimal notation and use A to denote the digit 10) and $2^{10}3^4 = 82944$. Are there other numbers with this property?

Theorem 1. *If $1024 \cdot 3^c - c$ is divisible by 10 for some integer $c \geq 0$, then $1024 \cdot 3^c$ is a Meertens number in base $b = \frac{1024 \cdot 3^c - c}{10}$.*

Number base	Meertens number
2	2, 6, 10
3	10
4	200
5	6, 49000, 181500
6	54
7	100
8	216
9	4199040
10	81312000
14	47250
16	18
17	36
19	96
32	256
51	54
64	65536
71	216
158	162
160	324
323	1296
481	486
512	4294967296
1452	1458
1455	2916
1942	5832
4096	65536
4367	4374
7775	46656
8294	82944
13114	13122
13118	26244
26242	104976
39357	39366
52485	157464
74649	746496
118088	118098
209951	1679616
354283	354294
1062870	1062882
1119743	10077696

Table 1: Meertens numbers in various number bases.

Proof: First note that $b > 10$, $b > c$ and $1024 \cdot 3^c = 10b + c$ written in base b has digits 10 and c which maps to $1024 \cdot 3^c$ under the map M. □

There are two solutions with $c < 10$, i.e, $c = 4$ and $c = 6$, with $c = 4$ corresponding to the number 82944 above and $c = 6$ corresponding to a Meertens number 746496 in base 74649. Similarly, $2^{100}3^{96} - 96$ is a Meertens number in base $\frac{2^{100}3^{96}-96}{100}$ and the base in decimal is equal to the Meertens number in decimal minus the last 2 digits.

Since M is not injective, it is possible for a number to be a Meertens number in more than one number base. We note in Table 2 that 6, 10, 216 and 65536 are Meertens numbers in more than one number base. Are the any others? The answer is yes as a consequence of the following result.

Theorem 2. *If a, k and m are positive numbers such that $a + km = 2^a$ and $a < k$, then 2^{2^a} is a Meertens number in base 2^k. In particular for $a > 2$, 2^{2^a} is a Meertens number in base 2^{2^a-a}.*

Proof: Since $2^{2^a} = 2^a 2^{km}$, this means that 2^{2^a} consists of a single digit of value $2^a < 2^k$ followed by m zeros. Thus $M\left(2^{2^a}\right) = 2^{2^a}$. For $a > 2$, $2^a - a > a$ and by setting $m = 1$, this shows that 2^{2^a} is a Meertens number in base $2^{2^a} - a$. □

In particular, if $k > a$ is a divisor of $2^a - a$, then 2^{2^a} is a Meertens number in base 2^k. For small values of a we list these divisors in Table 2.

a	k: divisors of $2^a - a$ larger than a
3	5
4	6, 12
5	9, 27
6	29, 58
7	11, 121
8	31, 62, 124, 248
9	503
10	13, 26, 39, 78, 169, 338, 507, 1014
11	21, 97, 291, 679, 2037
12	1021, 2042, 4084
13	8179
14	1637, 3274, 8185, 16370
15	4679, 32753

Table 2: Values of a and k such that 2^{2^a} is a Meertens number in base 2^k.

This shows that there are many other numbers (such as $4294967296 = 2^{2^5}$) that are Meertens numbers in more than one base. For instance $2^{2^{16}}$ is a Meertens number in at least 105 different bases and $2^{2^{64}}$ is a Meertens number in at least 435 bases! In particular, for $t > 2$, $2^{2^t-k} - 2^{t-k}$ is a divisor of $2^{2^t} - 2^t$ for $k = 0, \cdots, t$. Thus $2^{2^{2^t}}$ is a Meertens number in at least $t + 1$ different bases, i.e. there are numbers which are Meertens numbers for arbitrarily large number of bases. Even though there is only one known Meertens number in base 10, the above also implies that there are arbitrarily large bases for which Meertens numbers exist.

Theorem 3. *For a positive number n, $2 \cdot 3^n$ is a Meertens number in base $2 \cdot 3^n - n$.*

Proof: Since $n < 2 \cdot 3^n$, $2 \cdot 3^n$ is written as $1n$ in base $2 \cdot 3^n - n$, and $M(2 \cdot 3^n) = 2 \cdot 3^n$. □

3 Injective Gödel-like encodings

As mentioned earlier, the encoding defined by $M(m)$ is not a proper Gödel encoding as it is not one-to-one. Next we look at some injective Gödel-like encodings.

3.1 M_2 Meertens number

Consider the one-to-one function $M_2(m) = \prod_{i=1}^{n} p_i^{d_i+1}$. We will call numbers such that $M_2(m) = m$ a M_2 Meertens number. If the encoding is one-to-one, there cannot be a number n that is a fixed point of this encoding in more than one basis. This is easily seen as a number will have different digits in different bases. Some examples of M_2 Meertens numbers in various bases are listed in Table 3.1.

base	M_2 Meertens number
12	12, 24
16	48
24	96
35	36
64	384
106	108
107	216
115	576
192	1536
321	324
329	2304
431	1296
968	972
970	1944
1943	7776
2048	24576
2911	2916
8742	8748
8745	17496
11662	34992
24576	393216
26237	26244
46655	279936
78724	78732
78728	157464
157462	629856
236187	236196
314925	944784

Table 3: M_2 Meertens numbers in various bases.

The following result shows that there are an infinite number of M_2 Meertens numbers.

Theorem 4. *For $t \geq 0$, $3 \cdot 2^{2^t+1}$ is a M_2 Meertens number in base $3 \cdot 2^{2^t-t+1}$.*

Proof: First note that $2^t < 3 \cdot 2^{2^t-t+1}$. Then $3 \cdot 2^{2^t+1}$ in base $3 \cdot 2^{2^t-t+1}$ is the digit 2^t followed by the digit 0 which maps to $3 \cdot 2^{2^t+1}$ under the mapping M_2. □

On the other hand, for a fixed b, there are only a finite number of M_2 Meertens numbers in base b.

Definition. Let p_n denote the n-th prime number, Let $p_n\#$ denote the primorial defined as $p_n\# = \prod_{i=1}^{n} p_i$. Let $\nu(t)$ denote the first Chebyshev function defined as $\nu(n) = \sum_{p \leq n} \log(p)$ where p ranges over all prime numbers less than or equal to n.

Theorem 5. $p_n\# > n^{0.5972n}$.

Proof: For $n = 1$, the statement is trivially true. For $n > 1$, note that $p_n\# = e^{\nu(p_n)}$. Rosser [3] showed that for $n \geq 1$, $p_n > n \log n$. In [4], it was shown that for $n \geq 41$, $\nu(n) > n(1-1/\log(n)) > 0.73n$. For primes $2 < p_n < 41$, numerical computation shows that $\nu(p_n) > 0.5972p_n$. This implies that $p_n\# > e^{0.5972p_n} > e^{0.5972n \log n} = n^{0.5972n}$ for $n > 1$.

Theorem 6. *If m is a M_2 Meertens number in base b with k digits, then $b^k > 2p_k\#$.*

Proof: Since m expressed in base b has k digits, $m < b^k$. On the other hand, $m = M_2(m) \geq 2p_k\#$. □

Theorem 7. *If m is a M_2 Meertens number in base b, then $m < b^{b^{1.675}}$.*

Proof: Suppose that m expressed as a base b number has k digits. Then by Theorem 6 and Theorem 5, $b^k > 2p_k\# > k^{0.5972k}$, implying that $k < b^{1.675}$. Thus $m < b^k < b^{b^{1.675}}$. □

Corollary 1. Let k^* be the largest integer k such that $b^k > 2p_k\#$. Then $k^* \leq b^{1.675}$. If m is a M_2 Meertens number in base b, then $m < b^{k^*}$.

Proof: This is a consequence of Theorem 6 and Theorem 7. □

Corollary 2. For $b \leq 1000$, if m is a M_2 Meertens number in base b, then $m < b^{b-1}$.

Proof: This requires a computer-assisted proof by computing the value of k^* in Corollary 1 for various b. □

We conjecture that Corollary 2 is true for all b. In particular, a plot of k^* versus b is shown in Fig. 1.

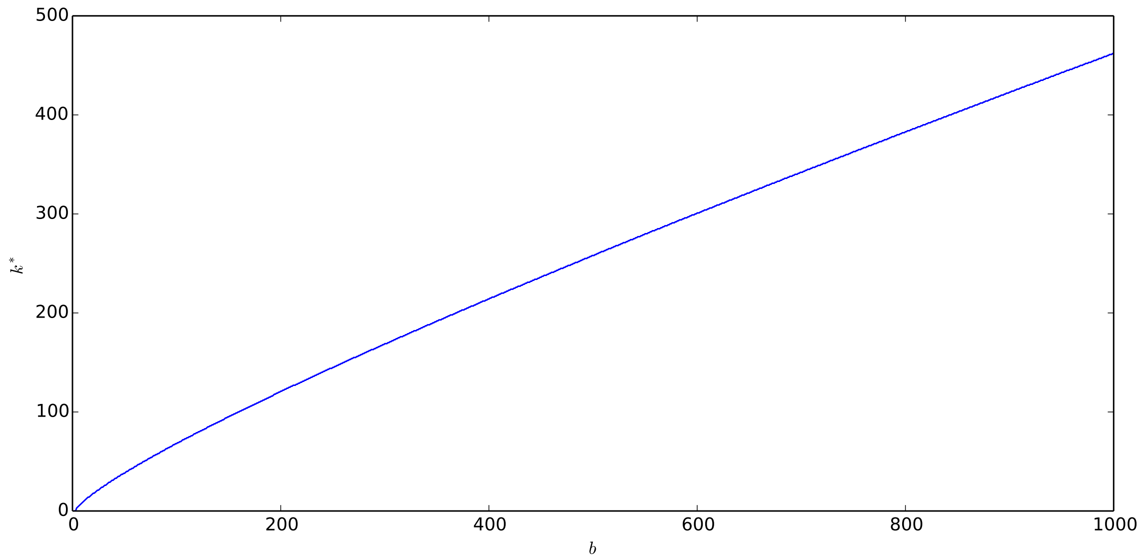

Figure 1: The value of k^* as defined in Corollary 1 as a function of b.

The following result shows that 12 is the smallest base for which there exists a M_2 Meertens number.

Theorem 8. *There are no M_2 Meertens numbers in base < 12.*

Proof: This again requires a computer-assisted proof. If m is an M_2 Meertens number in base b, then Corollary 2 implies that $m \leq b^{b-1}$. Next an exhaustive search up to b^{b-1} for $b < 12$ shows that there are no M_2 Meertens numbers in base < 12. □

3.2 Reverse Meertens number

Another way to define a one-to-one encoding is by reversing the digits and applying M, i.e. if the base-b representation of a number m is $d_n \cdots d_1$, then the encoding $M_r(m) = \prod_{i=1}^{n} p_i^{d_i}$. As before, because this encoding is one-to-one, a number can be a reverse Meertens number in at most one number base. In base 10, $12 = 3^1 2^2$ is a reverse Meertens number. Reverse Meertens numbers in different bases are listed in Table 3.2.

Note that 17496 is both a reverse Meertens number and a M_2 Meertens number (in different bases).

Theorem 9. *$p_{r+1}^{p_{r+1}}$ is a reverse Meertens number in base $p_{r+1}^{\frac{p_{r+1}-1}{r}}$ if r divides $p_{r+1}-1$.*

Proof: Since $k < k^i$ for $k, i > 1$, consider a base k^i representation consisting of the digit k followed by r zeros, where $r = \frac{k-1}{i}$. This represents the number $m = k(k^i)^r = k^{ir+1} = k^k$. Under the mapping M_r, $M_r(m) = p_{r+1}^k$. Then the result follows if $k = p_{r+1}$. □

In particular, some values of p_{r+1} satisfying the condition in Theorem 9 are: 3, 5, 7, 31, 97, 101, 331, 1009, etc.

4 Generalized Meertens numbers and generalized reverse Meertens numbers

Definition. Given maps $f = \{f_1, f_2\}$ where $f_1 : \mathbb{N} \to \mathbb{N}$, $f_2 : \mathbb{N} \to \mathbb{N}$, define the map

$$M_f(d_1, \cdots, d_n) = \prod_{i=1} f_1(i)^{f_2(d_i)}$$

A generalized Meertens number (GMN) in base b is a number m such that $M_f(d_1, \cdots, d_n) = m$ where $(d_1, \cdots, d_n)$ are the digits of m in base b. A generalized reverse Meertens number (GRMN) in base b is a number m such that $M_f(d_n, \cdots, d_1) = m$.

In the cases we discussed above, $f_1(i)$ is the i-th prime. For these cases, since $p^d > d$ for all primes p and integers d, all GMN and GRMN in base b must be larger or equal to b. The tables above show that it it possible for a GMN or GRMN in base b to be equal to b. In particular, 2 is a Meertens number in base 2, 12 is a M_2 Meertens number in base 12 and 3 is a reverse Meertens number in base 3. In fact, since b written in base b is 10, applying the digits $(1, 0)$ to M_f will return a number b which is a GMN in base b. This is summarized in the following result.

Theorem 10. *Suppose $f_1(i) > i$ for all i. If a is a GMN or a GRMN in base b, then $a \geq b$. If $f_1(1)^{f_2(1)} f_1(2)^{f_2(0)} > 1$, then b is a GMN in base b where $b = f_1(1)^{f_2(1)} f_1(2)^{f_2(0)}$. Similarly, if $f_1(2)^{f_2(1)} f_1(1)^{f_2(0)} > 1$, then b is a GRMN in base b where $b = f_1(2)^{f_2(1)} f_1(1)^{f_2(0)}$.*

Consider the case where f_1 and f_2 are both the identity map, i.e. $f_1(i) = f_2(i) = i$. Clearly 1 is a GMN and a GRMN in this case. In base 10, $324 = 1^3 2^2 3^4$ is a GMN and $64 = 2^6 1^4$ is a GRMN. Table 4 lists some GMN and GRMN numbers under these f_i's.

References

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[4] J. Barkley Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois Journal of Mathematics **6** (1962), 64–94.

[5] Alan Turing, *On computable numbers, with an application to the entscheidungsproblem*, Proceedings of the London Mathematical Society **42** (1937), no. 2, 230–265.

base	reverse Meertens number
3	3, 10, 273
5	6, 175
7	100
9	27
10	12
17	36
21	24
25	3125
44	48
49	823543
70	144
71	216
91	96
97	486
186	192
194	972
285	576
323	1296
377	384
574	1728
760	768
1148	2304
1527	1536
2187	19683
2499	17496
3062	3072
4603	9216
4605	13824
5182	20736
6133	6144
7775	46656
9997	69984
12276	12288
12440	62208
18426	36864
24563	24576
36860	110592
49138	49152
73721	147456
98289	98304
209951	1679616
1119743	10077696

Table 4: Reverse Meertens numbers in various bases.

base	generalized Meertens number	base	generalized reverse Meertens number
2	1350	2	2,6,12
4	108	3	120, 360
5	8	4	54
6	16	5	48
7	72	6	32
10	324	7	768
12	1458	8	216, 1728
23	1728	10	64
29	64	11	192, 729, 1536

Table 5: Generalized Meertens numbers and generalized Meertens numbers in various bases for the case when f_1 and f_2 are identity maps. 1 is omitted from this table.