

IBM Research Report

A Practical Price Optimization Approach for Omni-Channel Retailing

Pavithra Harsha, Shivaram Subramanian, Markus Ettl

IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598 USA



Research Division

Almaden – Austin – Beijing – Brazil – Cambridge – Dublin – Haifa – India – Kenya – Melbourne – T.J. Watson – Tokyo – Zurich

A Practical Price Optimization Approach for Omni-channel Retailing

Pavithra Harsha

IBM T. J. Watson Research Center, Yorktown Heights, NY 10598, pharsha@us.ibm.com

Shivaram Subramanian

IBM T. J. Watson Research Center, Yorktown Heights, NY 10598, subshiva@us.ibm.com

Markus Ettl

IBM T. J. Watson Research Center, Yorktown Heights, NY 10598, msettl@us.ibm.com

Consumers are increasingly navigating across sales channels to make purchases. The common retail practice of pricing channels independently is unable to achieve the desired profitable coordination required between channels. As part of a joint partnership agreement with IBM Commerce, we engaged with three major retailers over two years, and developed advanced omni-channel pricing (OCP) solutions that are used by several retail chains today. A big-data platform is employed to develop an omni-channel framework to model location-specific cross-channel demand interactions. An integrated OCP optimization formulation profitably coordinates prices for non-perishable products across channels and store locations, taking into account the impact of competition, and sales goals. The resultant non-linear model is non-convex and NP-hard, and practically efficient optimization approaches are prescribed, along with computational results using real-world data. In the absence of certain side constraints, we derive insightful results on price coordination across channels.

An OCP implementation for a large retail chain yielded a 7% profit lift. IBM Commerce deployed proprietary versions of these models into production in 2014. Subsequently, IBM attributed the opening of several new market opportunities as well as significant incremental revenue to the deployed solution. In 2015, IBM formally recognized this work as a significant research accomplishment.

Key words: Omni-channel, pricing, attraction demand, regular pricing, cross-channel effects, elasticity

1. Introduction

The face of retail is changing with consumers increasingly navigating through multiple channels with ease to make purchases. Using smart phones, consumers shopping at a brick-and-mortar store can simultaneously visit the mobile or web store, or even the social networking storefront of the same retailer and its competitors to finalize a purchase. *Omni-channel retailing* is all about ensuring a seamless customer experience across all possible touch points, including stores, catalog, website, mobile, and social. It is aimed at revolutionizing how retailers engage with customers along every step of the consumers path to purchase, starting from product research to the final

purchase and beyond, including the ‘last-mile’ related to product delivery and consumer feedback. For example, omni-channel retailing includes (a) the opening of newer channels (physical stores, dot com websites, mobile or social networking pages) that consumers can access; (b) the employment of a variety of marketing measures that follow the customer seamlessly across channels to maximize their purchasing decision; (c) the use of advanced order fulfillment practices that enable a retailer to initiate ship-from-store fulfillment for e-commerce orders for faster delivery times and fewer out of stock situations; and last but not the least, (d) the buy-online-pick-up-in store offering.

Omni-channel retailing is a sweeping trend across the industry (Huffington Post 2013, Brynjolfsson et al. 2013, Bell et al. 2014) with the top retailers aggressively pursuing a variety of these strategies to attract the modern consumers and be well-positioned to maximize benefit from the accelerating mobile shopping trend. For example, an e-commerce platform vendor Shopify provides over 200K merchants with new sales channels by adding ‘Buy’ buttons through agreements with Facebook and Twitter (Shopify 2015). Retailers are shifting their advertisement spending from traditional to digital channels and the ad revenue growth is forecast to be 11% in the digital channels compared to 0.4% in the traditional channels (Business Insider 2015). The story for 2015’s Black Friday sales is the blurring of the line between in-store and the online shopping experiences with many large retail chains, like Target and Walmart, beginning to take an integrated approach to merchandizing (Retail Dive 2015).

Many of today’s large retailers started as single channel retailers and their supply chain was designed to ensure maximum efficiency and scale in that channel. These retailers operated either the brick-and-mortar channel or the e-commerce/online channel, which broadly encompasses all the digital/virtual channels such as websites, mobile and social. These retailers subsequently opened additional sales channels, and supported common and channel-specific assortments, in order to increase their customer base. However, these channels largely operated independently of each other in ‘silos’, with limited transparency and data sharing even within the organization. From the perspective of operations research technologies, many retailers today maintain separate brick and online merchandizing divisions, and employ decision support tools for demand forecasting, pricing optimization, and inventory management that are channel specific, often procured from different vendors. Such tools largely ignore the multi-channel shopping path of today’s customers, as well as the potential efficiencies of integrated omni-channel decision support systems. Recently the Walmart CEO Doug McMillon said, “I want us to stop talking about digital and physical retail as if they’re two separate things. The customer doesn’t think of it that way, and we can’t either,” in the annual shareholders meeting (WAL 2015).

The omni-channel environment is also characterized by a highly competitive and dynamic marketplace due to the presence of e-tailers whose product offerings are *price-transparent* because of

comparison shopping websites sites such as *shopsavvy*, which exert a downward pressure on the retailers sales and profitability. Moreover, the online marketplace is witnessing a steep growth rate even as brick stores are transforming their operations to attract more walk-in customers and make in-store purchases more viable. The internet retailer magazine reports that in 2014 online sales grew six times faster for U.S. top 500 merchants than total retail sales, at 16% compared to 2.4% ([Internet Retailer 2015](#)). A key challenge here for retailers is to not cannibalize, but preserve and enhance the viability of traditional channels over time, while also increasing their online presence.

Given these emerging challenges in the omni-channel market-place, retailers are attempting to fundamentally transform their business models and organization structure, and synchronize the merchandizing decision systems supporting their sales channels in order to be more nimble, integrated, and effective at being customer centric, and profitable. This work is focused on developing a novel solution that overcomes some of these challenges by integrating key decisions, specifically pricing, across the different sales channels.

Consider for example, a retail chain that operates two sales channels (say brick-and-mortar, and online), and wants to set prices for products sold in these channels. A mixture of customer segments across various demographics including traditional walk-in customers, as well as mobile-enabled, internet-savvy customers purchase products from the retailer. Keeping this in mind, the brick prices across the store locations and the online prices that are set by the retailer have to be profitable while also meeting sales goals and being competitive with large e-tail giants who are steadily gaining market-share. The retailer's online price should not be cannibalizing brick sales at these store locations and vice-versa. Given this, the retailer is faced with the question of whether to match prices across channels or not, at what price levels, at what location, and for which products. With the increase in the number of digital channel offerings in the future and dynamic nature of the marketplace, the scale and speed of these decisions become critical. Current retail revenue management and pricing decision systems are limited by their capabilities, which were largely built to support an earlier, single-channel world of either brick or e-commerce. They are unable to manage the diversity and volume of data, or possess the advanced analytical techniques that are needed to effectively answer such questions, sometimes in (near) real-time. The integral omni-channel pricing solution that we have proposed and developed for the retail industry in this paper addresses all these questions.

As part of a joint partnership agreement with our client, [IBM Commerce](#), a leading provider of merchandizing solutions, we engaged with three major omni-channel retailers over a period of two years, who were faced with one or more of the challenges described above. Our engagement with the retailers was from the perspective of a retail analytics solution provider, in that our goal

was to develop advanced omni-channel retail analytics solutions for commercial use by current and future retail customers of IBM Commerce, keeping in mind the infrastructure and operational requirements of a deployable solution.

We now summarize the main contributions of this paper:

- 1. Omni-channel demand modeling and optimization framework:** We design and develop an omni-channel demand modeling framework over which a suite of advanced omni-channel retail analytical solutions such as demand forecasting, pricing, and inventory management can be built upon. Within this framework, we can predict demand at the location-specific channel level, providing the flexibility to model sales channels as purchase choices for consumers in a location, while also capturing the heterogeneity of channel preferences across locations. This method steps away from our client's traditional approach of viewing the online channel as a single entity independent of the brick channel (e.g., yet another store location of the retailer).
- 2. Omni-channel price optimization:** We study the omni-channel pricing (OCP) of non-perishable products geared towards regular pricing solutions (also commonly referred to as base pricing or everyday pricing). Our method gainfully employs the aforementioned demand modeling framework to formulate and solve an integrated price optimization problem across multiple channels and locations. In this paper, the OCP problem for a non-perishable product is practically motivated with business constraints on prices that maximizes the retailer's gross profitability across all locations and channels, while also satisfying certain volume and price-image goals. We use attraction demand models to represent consumer choice across various channels and observe that the resulting optimization model is a non-linear, non-convex NP-Hard problem. In the absence of certain side constraints, we show that the optimal price online is a weighted average of the optimal brick prices in various locations plus a known constant that depend on the price sensitivity coefficients of the two channels. In general, we propose a pseudo polynomial decomposition based approach to solve this pricing problem in the presence of two channels such as the brick-and-mortar and online channels. In the case of two or more channels, we employ specialized mathematical transformations to recover a mixed-integer programming (MIP) reformulation that can be solved efficiently using commercial off-the-shelf MIP solvers such as IBM ILOG CPLEX.
- 3. Implementation and business value assessment:** We share the in-depth results of the business value assessment conducted as a part of our OCP implementation for one of the major omni-channel retailers in the United States. For 100 products in the two product categories that we analyzed, we observed that the cross-channel price effects were significant, and vary by category, and can be as high as 50% of the own channel price elasticity. We solved the resulting

omni-channel pricing formulation to optimality, and obtain a 7% profit lift using omni-channel pricing over their existing channel-independent pricing methods. These results were presented at the retailer's site to a team that included senior executives, and their Vice-President for revenue management. Their response was overwhelmingly positive, and with similar experiences with other retailers, our proposed solution was approved for commercial deployment.

Commercial Success: Proprietary versions of the models presented in this paper were deployed into production by IBM Commerce in 2014 as a cloud solution. This solution was showcased as one of the retail analytics success stories in the smarter-commerce global summit in 2014. Today, several large global retail chains are regular users of the commercial offering including those with whom we engaged. Overall, IBM has directly attributed several new market opportunities as well as significant incremental revenue to IBM in 2015 due to the deployed OCP solution. In November 2015, this work was formally recognized by IBM as one of the major accomplishments in 2015 by the research division.

Organization: The remainder of the paper is structured as follows. In [Section 2](#) we review related literature. In [Section 3](#) we describe the omni-channel demand modeling and optimization framework. We discuss the parametric form of the omni-channel demand model we use in [Section 4](#), and in [Section 5](#), we formulate the resultant price optimization problem and discuss its tractability. In [Section 6](#) we describe prescriptive optimal conditions on price coordination between the different channels, followed by solution methods in [Section 7](#). In [Section 8](#) we discuss the OCP implementation and the business value assessment presented to the retail customer on their data. We conclude in [Section 9](#) with a brief discussion of data flow in the commercial deployment along with other practical use-cases of OCP, lessons learnt, and our some post-deployment highlights

2. Literature Review

Omni-channel retailing is a relatively recent phenomenon in the retail industry. Therefore, not surprisingly, there are not many papers in the available literature that study this topic, with little or discussion about its operations aspect.

Studying and modeling the consumer preferences in an omni-channel environment and understanding shift from the previous single-channel era is a first step that can pave its way into operations. Some recent papers in the marketing literature have explored consumer dynamics in a multi-channel environment, in particular, consumers migration across multiple channels ([Ansari et al. 2008](#), [Chintagunta et al. 2012](#)) and the impact on cross-channel elasticities when newer channels are introduced ([Avery et al. 2012](#)). The former papers use binary choice probit models to calibrate the substitution behavior of consumers between two channels (web and catalog or online

and brick respectively). More recently, (Gallino and Moreno 2014) empirically study the impact of sharing inventory information on the channel sales in the context of a buy-online-ship-from-store offering by a retailer. In this paper, we estimate consumer channel preferences from data, similar to the consideration in the above papers, using attraction demand models (McFadden 1974, Urban 1969) and geared to improve operational pricing decisions in the presence of two or more channels.

Game theoretic analysis to study price competition in a multi-channel context has been a topic of recent interest. Kireyev et al. (2014) study how and when self-matching prices can be an effective pricing strategy for the multi-channel retailers in the presence e-tailers and other multi-channel retailers. Hua et al. (2010), Yan and Pei (2011) study competition due to the pricing policies in a supply chain with a retailer and a separate multi-channel manufacturer (with online and traditional retail channels).

From an operational perspective, there is substantial academic literature that focus on single and multi-product pricing problems (for example, see the survey papers by Bitran and Caldentey 2003, Elmaghraby and Keskinocak 2003, Chen and Simchi-Levi 2012). To the best of our knowledge, the focus has been on single channel pricing and largely restricted to single location. Commercially available pricing solutions employed by retailers today price the multiple channels of a retailer separately or sequentially. In contrast, the focus of this paper and the deployed solution is on an integrated multi-channel and multi-location pricing problem in the presence of cross-channel demand interactions and important practical considerations.

From the perspective of price optimization using customer choice models, several papers in the literature have analyzed a variety of parametric and non-parametric approaches. For the multinomial logit (MNL) demand model, Hanson and Martin (1996) show that the profit as a function of the prices is not *quasi-concave*. Aydin and Porteus (2008), Akçay et al. (2010) explored this problem further and show that the resultant profit function is unimodal in the price space. Meanwhile, Song and Xue (2007), Dong et al. (2009) proposed a market share variable transformation to demonstrate that the objective function is jointly concave in the space of the market share variables. This transformation idea for MNL demand models was later extended to general class of attraction models by Schön (2010), Keller et al. (2014).

Our work more specifically relates to the pricing with mixture of attraction demand models. For a mixture of attraction demand models, the pricing problem is an open problem as pointed by Keller et al. (2014), who develop a local optimal heuristic solution by employing an approximate demand model and assuming convexity. On the other hand, for the non-convex problem we analyze and we develop tractable exact solution methods to solve large-scale OCP problem instances that arise in practice. In Section 9.1, we comment on the unsuitability of suboptimal approaches for practical deployment as they can often lead to incorrect downstream decisions.

A few papers have explored the use of other demand models in the context of multi-item pricing problems. For example, pricing nested logit demand models has been studied by [Li and Huh \(2011\)](#), [Gallego and Wang \(2014\)](#), [Rayfield et al. \(2012\)](#). In the presence of varying price elasticities across items, and unconstrained prices, [Gallego and Wang \(2014\)](#) show that the resultant problem can be computationally intractable as a transformed model is non-convex. For multi-item pricing using a hybrid MNL demand model, [Subramanian and Sherali \(2010\)](#) provide a mixed-integer linear programming formulation that incorporates a variety of practical business rules by simultaneously working in the price and market share space. Non-parametric approaches to multi-item pricing have been explored by [Rusmevichientong et al. \(2006\)](#) and [Aggarwal et al. \(2004\)](#) using heuristic approaches and approximation algorithms.

3. Omni-Channel Demand Modeling and Optimization Framework

In this section, we present the design of a framework over which a suite of advanced omni-channel retail analytical solutions such as demand forecasting, pricing, and inventory management can be built upon. The OCP problem developed in this paper uses this framework.

Consider an omni-channel retail chain that wants to understand its customers channel preferences and propensity to purchase. The existing demand forecasting systems (and that of our client) use historical point-of-sales data for the online store and the brick stores along with the associated sales attributes such as price, promotion, seasonality, etc., to calibrate separate demand models for every store cluster as a function of these attributes. In particular, the online channel is treated as an independent single-store cluster.

On the other hand, as shown in [Fig. 1\(a\)](#), the retail chain's consumers navigate across channels and retailers to finalize a purchase. The incumbent demand forecasting approach limits the retailer's ability to accurately quantify this behavior of omni-channel customers. For example, while this approach is quite capable of identifying the effect of the online price on brick sales at any physical store location, it is difficult to quantify the impact of brick-store prices on online-channel sales. Mathematically, the existing system can calibrate a reasonable model for demand at location B_j denoted by $D_{B_j}(p_O, p_{B_j})$ where p_O is the online price, and p_{B_j} is the brick price at location B_j , whereas it is not easy to estimate the online demand denoted by $D_O(p_O, p_{B_1}, p_{B_2}, \dots)$ as a function of all the brick prices. The sheer number of physical store locations (ranging from several hundreds to a few thousand), and the usage of location-specific pricing makes this task impractical. While employing some store-weighted average brick price across locations may simplify the parameter estimation procedure, the resultant loss in sensitivity to location specific brick prices, store promotions, and the impact of local events precludes the accurate quantification of the a brick location's cross channel impact on the online channel. This inability to quantify cross-channel sales

impact across the retail chain would result in sub-optimal decisions being produced by downstream decision systems. We overcome this limitation by designing an alternative framework that enables retailers to model a variety of cross-channel interactions benefiting both the demand, as well as the supply side.

In Fig. 1(b) we propose a demand modeling framework for the omni-channel environment to accurately quantify cross-channel demand interactions. We partition the online store (transactions of which, in reality, originate from a continuum of customer zip-codes) into multiple virtual online stores (virtual stores, for brevity). Each virtual store corresponds to the online transactions originating from a specific geographical aggregation of zip-codes. These zip-codes can also represent the customer-base of the physical stores in that region, referred to as a brick store cluster. Thus, customers within the zip-codes associated with any virtual store, can choose to purchase from this virtual store or the brick store cluster in that area. Virtual store transactions whose zip-codes cannot be mapped to any physical store cluster in the vicinity are grouped into a single virtual store that we denote as O_o in the figure. In Fig. 1(b) we create one virtual store for every physical brick store. Often times, multiple brick stores can be located in neighboring zip-codes, and imputing the online transactions at the brick-store level is difficult. Therefore, identifying a suitable level of aggregation in the location hierarchy of the retailer to create the virtual stores is important so as to balance the richness of the predictive model and forecast accuracy. Here on, we use the term location or zone interchangeably to represent the level of geographical aggregation at which we develop the predictive model.

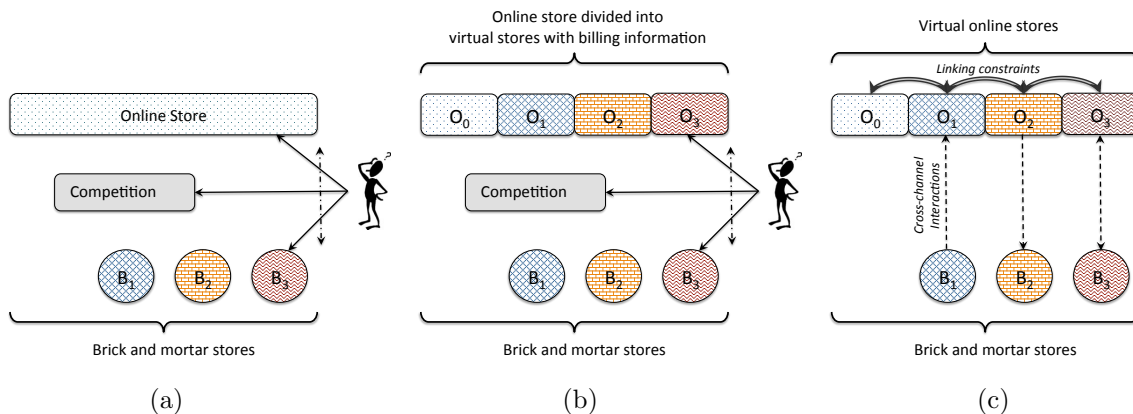


Figure 1 (a) Current retail demand modeling framework; (b) Proposed omni-channel demand modeling framework; and, (c) Illustration of the cascading interactions across channels and locations. The dotted lines represent the cross-channel effects and double solid lines represent the linking constraints.

The main advantages in creating these virtual online stores are as follows:

1. Calibrating the demand model for a virtual store reduces to solving a local parameter estimation problem without the need to consider the impact of all the (several hundred) location-specific prices or promotions. For example, from a pricing point of view, the demand at virtual store O_3 is just $D_{O_3}(p_{O_3}, p_{B_3})$. Here we assume that different physical store clusters have negligible interaction from a demand perspective which is quite reasonable. The total online demand is, therefore, the sum of demands of the virtual stores.
2. The demand modeling framework in Fig. 1(b) naturally lends itself to modeling the channel-switching propensity of customers within an omni-channel environment. Geographical clustering enables the demographic heterogeneity of online shoppers to be automatically incorporated. Finer levels of consumer segmentation are also possible within this framework. In fact, if customer loyalty card data is available, this framework can be used to calibrate customer-specific, personalized demand models.
3. Incorporating the effects of competition, product ratings, social network sentiment, and local events in demand modeling is useful, and is relatively easy to do within the proposed framework. Brick as well as online competitor prices, if available, can be introduced as additional attributes, or as additional customer choices in order to account for lost sales.
4. Our virtual store model enables downstream systems to accurately calculate shipping costs, and delivery times for online orders as well guide fulfillment decisions.

In Fig. 2 we provide an example of the geographical dispersion of the volume of sales (represented by the size of the pie) and channel share between brick (red) and online (blue) for a consumer electronics category of an omni-channel retailer. We clustered their store network (more than 1500 stores) using a k-means distance based algorithm (MacQueen et al. 1967) where $k = 50$. The figure shows the vornoi diagram created using the centroids of the clusters, which also enables the partitions of zip-codes into the zones. Interestingly, the figure shows that the channel preferences varies by location. Our clustering method not only captures the heterogeneity in the volume of sales but also the variation in the channel preferences of consumers. This visualization of the data was developed using D3 libraries and is a key output shared with retailer.

In the future, explicitly distinguishing between the virtual channels (e.g. mobile, social, video) may be essential because of the emerging differences in merchandizing strategies adopted. A similar approach demonstrated for the online channel can be performed for every virtual channel.

A big-data platform can be gainfully employed to manage the variety, volume, and velocity of data required to support and periodically update the omni-channel demand modeling framework within a practical application.

Integrated decision making across the retail chain: The proposed omni-channel framework enables a more accurate characterization of demand and can be employed to solve omni-channel

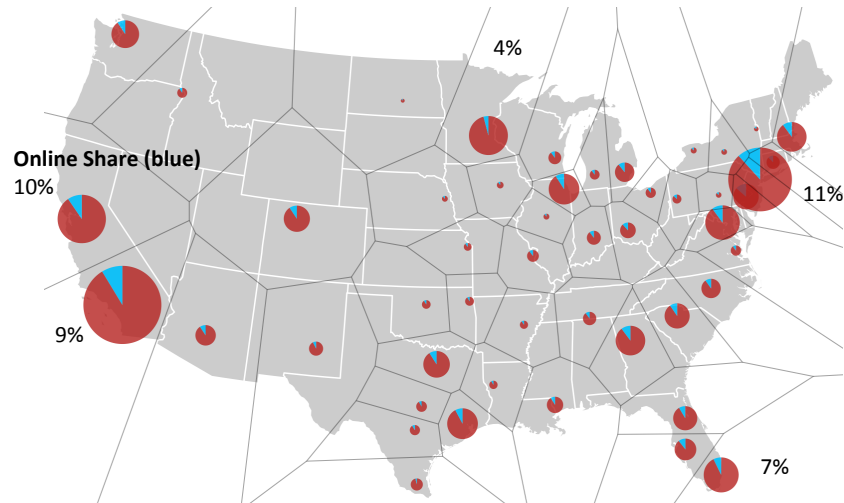


Figure 2 Example of the volume of sales and channel share across the geographical clusters for one electronics category of an omni-channel retailer.

decision problems such as a pricing and inventory management. Toward this, observe that the propensity of customers to switch channels based on price comparisons, as well as the natural pricing constraints between locations (e.g. a common price across virtual stores), and channels (e.g., online price must be within 20% of the brick price) results in a cascading, directed price-influence network, as depicted in Fig. 1(c). Therefore, in today’s omni-channel environment there is a need for integrated decision making across all channels and locations.

The benefits of integrated decision making across the retail chain are as follows: (1) it minimizes cross channel sales cannibalization, as well as lost sales to competitors; (2) it better manages inventory costs across the retail chain by modeling cross-channel fulfillment possibilities such as ship-from-store and buy-online-pickup-instore; (3) it better satisfies inter-channel and inter-location constraints (such as price-matching between channels or locations, and global sales goals) which are of practical importance; (4) the analysis can be readily extended to manage more than two channels, as well as multiple shipping tiers for online product purchases, and (5) the retailer is better positioned to profitably manage the increasing online sales presence in the future marketplace, where cross-channel demand interaction levels are likely to be even higher.

On the other hand, integrated decision making in an omni-channel environment requires us to solve a large-scale optimization problem spanning the retail chain, along both the channel, and location dimensions. In this paper, we highlight how certain reformulation and decomposition techniques help overcome this challenge in the specific context of prescribing optimal prices for a non-perishable (hardline) product sold in multiple channels and locations by a retailer.

We commenced the engagement with the retailers in the firm belief that with the plethora of data being collected, and the impressive strides made in large-scale optimization technologies over the years, such advanced revenue management solutions are not only practicable but also essential for an omni-channel retailer to retain the edge in today's competitive environment. The creation of the virtual stores and an integrated operation across the retail chain clearly makes the retailer's decision problems more customer centric (i.e., where does the demand originate from and what factors impacts demand) rather than being business- or operation-centric, as it was in the past.

4. Demand Model

Consider an omni-channel retailer selling a product using M sales channels to customers in J locations. Let $V \subset M$ be the set of virtual channels like website, mobile, social, which are partitioned into virtual stores by location $j \in J$. Let p_{jm} be the price for the product sold in channel $m \in M$ and location $j \in J$ and \mathbf{p}_j be the corresponding vector of prices in all channels at location j . Note that p_{jm} is often the same across $j \in J$ for virtual channels $m \in V$. Let $\mathbf{D}_j(\mathbf{p}_j)$ be the vector of demands originating from location $j \in J$ in all the channels. As motivated in [Section 3](#) we assume that the demand for a product in a specific channel and location depends on the attributes of all channels at that location in order to model customer choices across channels in an omni-channel world. We refer to this representation as the omni-channel demand model.

In deciding the specific class of the demand model, one has multiple alternatives depending on the features it captures as well as the practical ease of parameter estimation using historical sales data. *Attraction demand models* are one of the commonly used demand functions to model consumer choice in marketing, economics, and more recently, in the revenue management literature. They generalize the well-known *multinomial logit* (MNL) and the *multiplicative competitive interaction* (MCI) demand models, and have their foundations in the random utility theory in economics ([McFadden 1974](#), [Urban 1969](#)). We use these attraction demand functions to model consumer demand, more specifically the channel choice of a consumer, in an omni-channel environment. In particular, we assume it has the following form:

$$D_{mj}(\mathbf{p}_j) = \frac{\text{Market Size of location } j}{\text{Market Share of channel } m \text{ in location } j} \quad (4.1)$$

$$= \tau_j \frac{f_{mj}(p_{mj})}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} \quad (4.2)$$

where τ_j is the market size of location j and $f_{mj}(p_{mj})$ is the attraction function of customers in location j to channel m . The market size represents the measure of consumers interested in the product and the market share, also commonly referred to as the purchase/choice probability,

captures how consumers choose between different choices, including a no-purchase option. Some examples of the attraction demand model include the MNL demand model where $f_{mj}(p_{mj}) = e^{a_{mj} + b_{mj}p_{mj}}$, MCI demand model where $f_{mj}(p_{mj}) = a_{mj}p_{mj}^{b_{mj}}$, and a linear attraction demand model where $f_{mj}(p_{mj}) = a_{mj} + b_{mj}p_{mj}$. Here, a_{mj}, b_{mj} are constants that ensure the negative price elasticity of demand.

An attraction demand model is often chosen to model consumer choice even at an aggregate level (as we do in this paper) rather than just the willingness to buy of an individual because it has fewer coefficients to evaluate than its counterparts. In particular, the number of coefficients in the attraction demand model is $O(M)$ for M choices as opposed to $O(M^2)$ required in *scan-pro* demand models such as linear, log-linear or power-law models (Reibstein and Gatignon 1984, Berry 1994). Attraction demand functions also possess certain useful practical features that make it a viable demand modeling alternative for the downstream price optimization applications, as we see in Section 5.

The standard methods to estimate the parameters of an attraction demand model, and the MNL model in particular, are the maximum log-likelihood method (McFadden 1974, Ben-Akiva and Lerman 1985) and the ratio method (Berry 1994). Both methods require historical information about every choice. In our setting, this includes information about the historical transaction/sales data in every channel and location as well as *lost sales*. Lost sales refers to the instances where the no-purchase option is exercised by a consumer who finds all the available purchase choices to be less attractive. Mathematically it refers to the component $\tau_j \frac{1}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})}$ in the demand model in Eq. (4.2). Omni-channel retailers rarely have complete information about lost sales and more often than not, have to calibrate the attraction demand model using incomplete data.

In the seminal work on choice based revenue management by Talluri and Van Ryzin (2004), the authors develop an estimation method for this problem using an expectation maximization (EM) method. Newman et al. (2014) identify some drawbacks of the EM method and propose a computationally fast two step non-EM based estimation method. Talluri (2009) also describes a two-step risk-ratio based estimation method. We employ an alternative estimation method that we recently developed, and is described in our forthcoming paper (Subramanian and Harsha 2015). In that paper, we detail some of its advantages and provide a detailed computational comparison of the proposed method with other existing estimation methods on multiple real world data sets. However, any of these aforementioned estimation methods can be employed to calibrate our proposed omni-channel demand model. Also, during model calibration with real data, demand drivers besides price such as promotions, seasonalities, holidays and even competitor prices, if available, are included and we highlight this in our computational experiments in Section 8.2.

An alternative approach that is often employed in practice to model consumer choice and avoid estimating lost sales is the so-called *hybrid* attraction demand model (Subramanian and Sherali 2010). In these models, the market size is allowed to depend on the attributes of all the choices (e.g., prices in all channels). This modified market size denotes the set of people who purchase the product through one of the channels (i.e., it indirectly accounts for lost sales). The market share component is the standard attraction model only across the different choices available, excluding the lost share choice (i.e., without 1 in the denominator). The size of the pie in Fig. 2 represents the market size of a hybrid model and the channel fraction denotes the corresponding market share. Such a scheme simplifies the estimation procedure, which decomposes into two independent estimations of market size and share. Unfortunately, a hybrid model calibrated in this manner can yield counter-intuitive elasticity estimates (e.g., positive price elasticity for substitutive items), which are subsequently projected into the non-positive space after the estimation. Ex-post truncation can result in poor forecasts, whereas ignoring the sign of the price elasticity value can yield impractical pricing recommendations. Finally, the market size model injects additional nonlinearity and nonconvexity into a price optimization formulation as discussed in (Subramanian and Sherali 2010).

Attraction models satisfy the independence of irrelevant alternatives (IIA) property (McFadden 1974). As the number of (virtual) channels increases, the newer channels maybe perfectly substitutable and hence less likely to satisfy the IIA assumption. In such scenarios, alternative demand models that can overcome the IIA assumption, such as the nested attraction demand model, or the continuous mixture of attraction demand models (even at a single location) have to be considered and is a topic suitable for future research.

5. Omni-channel Price optimization (OCP) for a single non-perishable product

In this section, we formulate the omni-channel price optimization model for a non-perishable product in order to identify the most profitable prices in all channels and locations, subject to various retailer goals and practical business rules.

We restrict our analysis to the class of non-perishable items, and assume that there are well established replenishment policies, and that out-of-stock inventory effects negligible. This is a reasonable assumption for non-perishable goods. Mathematically, it allows one to view the integrated pricing problem across the retail chain as a single period pricing problem without inventory effects. For simplicity, we restrict our focus to the case when inter-item substitutive and complementary effects are absent.

Using the notation introduced earlier in [Section 4](#), we formulate the general non-linear omni-channel price optimization problem denoted by OCP as follows:

$$\text{OCP: } \max_{\mathbf{p}_j} \sum_{j \in J} (\mathbf{p}_j - \mathbf{c}_j)^T \mathbf{D}_j(\mathbf{p}_j) \quad (5.1)$$

$$\sum_j \mathbf{A}_{kj} \mathbf{D}_j(\mathbf{p}_j) \leq u_k \quad \forall k = 1, \dots, K \quad (5.2)$$

$$\sum_j \mathbf{B}_{lj} \mathbf{p}_j \leq v_l \quad \forall l = 1, \dots, L \quad (5.3)$$

$$p_{m,j} = p_{m,j'} \quad \forall m \in V, j, j' \in J \quad (5.4)$$

$$p_{mj} \in \Omega_{mj} \quad \forall m \in M, j \in J. \quad (5.5)$$

The decision variables in the above OCP formulation are the prices in all locations and channels, and the objective is to maximize the total profitability of the retailer across the retail chain. [Constraints \(5.2–5.3\)](#) are generic polyhedral constraints on demands and prices defined with known matrices $\mathbf{A}_k, \mathbf{B}_l \in \mathbb{R}^M \times \mathbb{R}^J$ and vectors $\mathbf{u} \in \mathbb{R}^K, \mathbf{v} \in \mathbb{R}^L$. These generic constraints encapsulate the retailer's goals and practical pricing rules. We provide several examples of these constraints in this section below. [Constraint \(5.4\)](#) ensures that the retailer offers the same price across all the virtual stores. This constraint is particularly relevant within our omni-channel framework because we explicitly partitioned the virtual channels by location in order to model cross-channel effects, and this constraint binds them back together from the view of the customer. Discrete pricing constraints, if present, are encapsulated in [constraint \(5.5\)](#).

Some examples of the generic business rules used in practice are as follows:

Volume (or sales goal) constraints

$$\sum_{m \in M_k, j \in J_k} D_{mj}(\mathbf{p}_j) \geq u_k, \quad (5.6)$$

where $M_k \subset M$ and $J_k \subset J$ and depending on the choice of M_k, J_k these constraints can be employed to support a retailers global or channel and location-specific sales goals. For example, [constraint \(5.6\)](#) can ensure that the total sales volume by channel does not drop below a user-specified threshold, u_k , thereby balancing profitability and market share objectives. Such constraints also act as a practical guard that prevent the drastic price increases that can occur while optimizing for weakly elastic products.

Price monotonicity constraints

$$p_{mj} \leq \gamma_{mm'} p_{m'j} + \delta_{mm'j} \quad \forall j \in J \text{ and for some } m, m' \in M. \quad (5.7)$$

The goal of [constraint \(5.7\)](#) is to enforce that prices in certain channels are cheaper than others by a specified percentage $\gamma_{mm'}$ and/or a constant $\delta_{mm'j}$. This constraint can also account for the variation in unit-cost across channels, i.e., the overhead cost of operating a physical store. An extension of [constraint \(5.7\)](#) is the *price-matching* constraint across the retail chain where the inequality is replaced by an equality and setting $\gamma_{mm'j} = 1$, $\delta_{mm'j} = 0$. Here, consumers can buy the same product anywhere in the retail chain at the same price. One can view [constraint \(5.7\)](#) also as a *volume measure* constraint. Sometimes a channel exclusively sells a larger volume measure or pack of the same product (for example, a 12-pack case of white board markers sold online versus a 6-pack case of markers sold in-store). Here, $\gamma_{mm'}$ is a scaling factor between channels that is employed to achieve price parity per unit measure.

Price bounds

$$\underline{\mu}_{mj} \leq p_{mj} \leq \bar{\mu}_{mj} \quad \forall j \in J, m \in M. \quad (5.8)$$

Here, $\underline{\mu}_{mj}$ and $\bar{\mu}_{mj}$ are upper and lower bounds that are often imposed as a percentage of historically offered prices or as a percentage of competitor prices to ensure the competitiveness of the retailer.

Discrete prices

$$p_{mj} \in \Omega_{mj} \quad \forall j \in J, \forall m \in M. \quad (5.9)$$

Magic number endings (e.g., those ending with 9) are important to a retailer and are encoded in this business rule. Furthermore, when a retail re-optimizes prices, [constraint 5.9](#) can be employed to generate a price ladder that proactively excludes trivial price changes in order to avoid the substantial labor cost incurred in physically changing the sticker prices in brick stores.

In practice, the choice of the business rules differ by product category and by the retailer. But in general, we classify all the business rules as either *inter-channel* or *inter-location* constraints, and treat restrictions on a specific channel-location pair as a subclass of inter-channel constraints. From a computational complexity perspective, inter-channel constraints are typically easy to satisfy (see [Remark 1](#) below for a counter example), whereas inter-location constraints tend to be harder. We discuss this issue in detail along with examples in the following subsection.

5.1. Special cases of the OCP and computational complexity

In this section, we analyze different special cases of the omni-channel price optimization problem and understand its computational complexity by positioning it in the context of current literature.

Single location and multiple channels: The OCP problem in this special case has a formulation that is structurally identical to the multi-item pricing problems with several side constraints.

Attraction demand models provide unimodal and convex structure to some special cases of the pricing problem using a market share transformation. We will explain the transformation below as we will use this later in the paper. The market share variables are defined as follows for the specific location j :

$$\theta_{mj} = \frac{f_{mj}(p_{mj})}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} \quad \forall m \in M, \text{ and} \quad (5.10)$$

$$\bar{\theta}_j = 1 - \sum_m \theta_{mj}. \quad (5.11)$$

Given the market share variables, there is a one-to-one transformation to the price variables as follows:

$$p_{mj} = f_{mj}^{-1} \left(\frac{\theta_{mj}}{\bar{\theta}_j} \right) = g_{mj} \left(\frac{\theta_{mj}}{\bar{\theta}_j} \right) \quad (5.12)$$

where $f_{mj}^{-1}(\cdot) = g_{mj}(\cdot)$ under the following very mild assumption on the structure of the attraction functions.

ASSUMPTION 1. (*Keller et al. 2014*) The attraction function, $f_{mj} : \mathbb{R} \rightarrow \mathbb{R}^+$ for each channel $m \in M$ and location $j \in J$ and satisfies:

1. $f_{mj}(\cdot)$ is strictly decreasing and is twice differentiable on \mathbb{R} , and
2. $\lim_{x \rightarrow -\infty} f_{mj}(x) = \infty$, and $\lim_{x \rightarrow \infty} x f_{mj}(x) = 0$.

Keller et al. (2014) provide a general condition under which the objective is concave function in the market share space. We state this condition as an assumption in our paper and in particular, the well know MNL, MCI and linear attraction demand models satisfy this assumption.

ASSUMPTION 2. (*Keller et al. 2014*) The function $g_{mj}(\cdot) = f_{mj}^{-1}(\cdot)$ satisfies the following condition for all $m \in M$ and the specific location j under consideration:

$$2g'_{mj}(y) + yg''_{mj}(y) \leq 0 \quad \forall y > 0. \quad (5.13)$$

The authors also show that the volume goals and price bounds [constraints \(5.6\)](#) and [\(5.8\)](#) respectively can be transformed into linear constraints in the market share space and therefore the multi-item problem with these constraints, under [assumption 2](#), can be solved efficiently as a convex optimization problem. However, this result does not extend in the presence of general linear pricing [constraints \(5.7\)](#) that are important in practical pricing applications. We motivate this in the following remark.

REMARK 1. The price monotonicity and the volume measure [constraints \(5.7\)](#) transform into highly non-linear and non-convex constraints in the market share space. For example, consider the

constraint $p_{mj} \leq \gamma p_{m'j}$ where γ is a constant. For an attraction demand model the constraint in the attraction space translates to $f_{mj}(p_{mj}) \leq f_{mj}(\gamma p_{m'j})$ which in the market share space translates to $\theta_{mj} \leq \frac{f_{mj}(\gamma p_{m'j})}{f_{m'j}(p_{m'j})} \theta_{m'j}$. Even for simple attraction models, the ratio of the attractions is not a constant. In the special case when $\gamma = 1$ and the b_{mj} are identical for all $m \in M$ in an MNL or linear attraction demand model, the right hand side can be transformed into an affine function in the market share space. We did not observe that b_{mj} are identical for all $m \in M$ for any of the product categories that we analyzed across retailers in our customer engagements. This can be attributed to the heterogeneity in people's shopping preferences at a location across different channels. Therefore, such a transformation fails to recover a convex formulation in either the price, or the market share space.

Multiple locations and multiple channels of a single type (virtual or not):

Suppose the multiple channels are physical store-like channels where prices can vary by location. In this case, the OCP problem decomposes by location in the absence of inter-location constraints, and reduces to multiple independent single-location, multi-channel pricing problems.

On the other hand, if the multiple channels under consideration represent virtual channels, the OCP problem has a structure that is identical to the pricing problem for a mixture of attraction demand models. Traditionally, such a mixture is across multiple segment classes where a customer has a certain probability of belonging to a particular segment. In the omni-channel modeling framework, a customer belongs to only one location, whereas the total demand is the sum of demands in each of these locations. A key feature of the mixture model is that the prices offered across the segments are the same for each choice, similar to the virtual channel prices in the OCP problem.

REMARK 2. In [Fig. 3](#) we plot the values of the objective function [Eq. \(5.1\)](#) for an OCP instance having a single virtual channel, say online, and two locations, with [constraint \(5.4\)](#) ensuring that the online price across locations is the same. We can observe from the figure that the objective function in this example is non-convex and has multiple peaks. Although [constraint \(5.4\)](#) is similar to the price matching constraint (see [Remark 1](#)), note that it is across locations and not choices. Because the lost sales probabilities, $\bar{\theta}_j$'s, vary across locations, the resulting constraint in the market share space is likely to inject a higher degree of non-linearity and non-convexity into the problem, when compared to the price matching constraint (which is a special case of [constraint \(5.7\)](#)).

We now show that the OCP problem having two or more virtual channels and just two locations is an NP-hard problem. This result is achieved by performing a reduction from the 2-class logit assortment optimization problem (2CL). The goal of 2CL is to identify an optimal assortment of

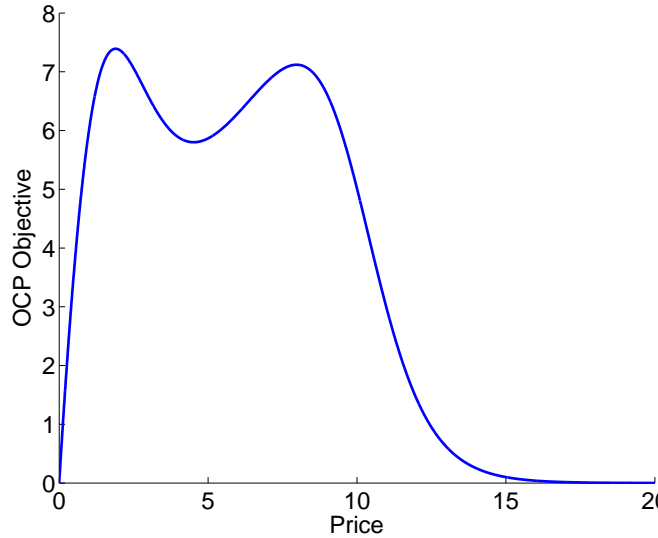


Figure 3 Example of a OCP objective for a single virtual channel and two locations as a function of the virtual channel price for an MNL attraction demand model with $a_{11} = 10, a_{12} = 1, b_{11} = 1, b_{12} = 1, \tau_1 = 1$ and $\tau_2 = 10$.

items in a set V to offer to customers who can potentially belong to one of two segment classes that are unknown to the seller. The inputs to this problem include the item profits, the relative weight of the classes and the preference weight of each item in each class. Rusmevichientong et al. (2010) showed that the 2CL is NP-hard. The reduction is as follows. Consider an arbitrary instance of 2CL and map every item in set V to a distinct virtual channels in the OCP problem. If an item $v \in V$ is part of an assortment then offer a product in the virtual channel v at some finite price that results in the attraction value of the channel-class being equal to the preference weight of the item in that class. If an item $v \in V$ is not part of an assortment then we offer the product in channel v at a sufficiently high price that reduces the attraction value of channel v to zero. The item profits in 2CL correspond to the margin of the channels, and the relative weight of the classes correspond to the market size of each location. As a reduction from this 2CL assortment optimization problem for an MMNL demand model, we can see that the OCP problem is NP hard. We state this as a remark below.

REMARK 3. *The OCP problem with multiple virtual channels and at least two locations is NP-hard.*

A relevant question for this paper is the complexity of the OCP problem having multiple locations but a limited number of channels (with one or more virtual channels). While this remains an open question, we present an empirically efficient pseudo-polynomial algorithm in this paper for the single virtual channel instance and propose a tractable MIP to manage the general case.

Multiple locations and multiple channels of different types: This setting encapsulates the OCP problem of interest. Clearly, the presence of the virtual channels and practical business

rules makes this case non-convex and non-linear. Solutions obtained using gradient-based, non-linear programming can be stuck in local optima which can be far away from the global optimum of the problem, resulting in poor quality pricing recommendations. Randomized meta-heuristic methods can be employed to partially overcome this problem. In [Section 9.1](#), we comment on the unsuitability of such suboptimal approaches for practical deployment. In the following sections, we study prescriptive conditions and exact solution methods to solve the OCP problem.

6. Price-coordination between channels

Price-coordination between channels for a product is often a retailer's choice and/or strategy that is based on the brand image to be maintained. Price-coordination can include, price-matching, or maintaining a certain price differential between channels. In this section, we provide a prescriptive solution for price-coordination using the parameters of the OCP attraction demand model. These solutions are meant to guide the retailer in the right direction when they set business rules and/or goals such as price-monotonicity. For this reason, we derive first order necessary conditions for the OCP problem while satisfying only [constraint \(5.4\)](#) that enforces the same price across virtual stores, and treat prices as continuous variables. The optimal prices in the presence of business rules can be obtained by using the methods described in [Section 7](#).

PROPOSITION 1. *For a retailer operating multiple channels at a single location, the optimal prices with a general attraction model satisfies the following condition:*

$$(p_o - c_o) - (p_b - c_b) = \frac{p_o}{\epsilon_o(p_o)} - \frac{p_b}{\epsilon_b(p_b)} \quad \forall o, b \in M \quad (6.1)$$

where $\epsilon_m(p_m) = \frac{-f'_m(p_m)p_m}{f_m(p_m)}$, the point price elasticity of the attraction function for channel m .

To provide more structure, we substitute the different forms of the attraction model and deduce the following corollary which we state without proof. We note that [Eq. \(6.1\)](#) is derived based on first order conditions and hence represents a necessary condition, but is not sufficient.

COROLLARY 1. *For an MNL demand model where $f_m(p_m) = e^{a_m - b_m p_m}$, [Eq. \(6.1\)](#) reduces to:*

$$p_o^* - p_b^* = c_o - c_b + \frac{1}{b_o} - \frac{1}{b_b} \quad \forall o, b \in M, \quad (6.2)$$

whilst for a linear attraction demand model where $f_m(p_m) = a_m - b_m p_m$, [Eq. \(6.1\)](#) reduces to:

$$p_o^* - p_b^* = \frac{1}{2} \left[c_o - c_b + \frac{a_o}{b_o} - \frac{a_b}{b_b} \right] \quad \forall o, b \in M. \quad (6.3)$$

On the other hand, for a power attraction demand model where $f_m(p_m) = a_m p_m^{-b_m}$ [Eq. \(6.1\)](#) reduces to

$$p_o^* \left[1 - \frac{1}{b_o} \right] - p_b^* \left[1 - \frac{1}{b_b} \right] = c_o - c_b \quad \forall o, b \in M. \quad (6.4)$$

A special case of the above corollary that is well-known from the multi-product pricing literature on MNL demand models is that if the price coefficients, b_o, b_b , are equal then so are margins at optimality (see e.g., Gallego and Wang (2014) and references therein).

Based on this corollary, we observe that for the MNL demand model in our OCP context, the optimal price gap between any two channels just depends only on b_m and is independent of the intercept a_m , and hence the channel market-shares. Also, if $b_o > b_b$ then $p_o^* - c_o < p_b^* - c_b$. In other words, if the online customers (o) are more price sensitive than the brick customers (b) then the optimal online margin is lower than the optimal brick margin. Moreover, if $c_o \leq c_b$ then at optimality $p_o^* < p_b^*$, i.e., the optimal online price is less than the optimal brick price. Finally because Eq. (6.1) holds for any two channels, the relationship between price-sensitivity and the optimal price holds for OCP instances with more than two channels as well. For example, if f refers to the social-networking store channel, then $p_f^* < p_o^* < p_b^*$, if we assume $c_f \leq c_o \leq c_b$ and $b_f > b_o > b_b$, i.e., the customers in the social channel are more price sensitive compared to the other online customers. These results extend directly to the power-law form of the demand model and the linear attraction demand models, except that the price coefficient in the latter case is b_m/a_m . We summarize this insight with the following remark.

REMARK 4. *If the cost $c_o \leq c_b$ for any $o, b \in M$ and if the price coefficient $\eta_o > \eta_b$ where η_m is b_m for the MNL and MCI demand models and is b_m/a_m for the linear attraction demand model then $p_o < p_b$.*

We obtain yet another insight from the above discussion and the corollary.

REMARK 5. *For a fixed set of costs across channels, as the gap between the price coefficients η_m increases (decreases), the gap between the prices at optimality increases (decreases).*

In other words, if widely different classes of customers shopping in the different channels, then the optimal channel prices can be sufficiently apart. On the other hand, if the customers who shop in the different channels become increasingly homogenous, their price sensitivities will be similar and the prices will be closer.

Finally, for the MNL demand model from Eq. (6.2), when all the channels are highly price sensitive, say $b_m > 1 \forall m \in M$, then the gap in the margins (or prices if the costs are equal across channels) are less than 1.

We now consider the case of a retailer operating across multiple locations.

PROPOSITION 2. *For a retailer operating multiple channels and multiple locations, let $w_j(\mathbf{p})$ be the normalized values corresponding to the online demand in each location, i.e., $\tau_j \frac{f'_{oj}(p_o)}{1 + \sum_{m' \in M} f'_{m'j}(p_{m'j})}$,*

over all the locations $j \in J$ resulting in $\sum_{j \in J} w_j(\mathbf{p}) = 1$. Then the optimal prices with a general attraction model satisfy the following condition:

$$\sum_{j \in J} w_j(\mathbf{p}) \left[(p_o - c_o) - (p_{bj} - c_{bj}) - \frac{p_o}{\epsilon_{oj}(p_o)} + \frac{p_{bj}}{\epsilon_{bj}(p_{bj})} \right] = 0 \quad (6.5)$$

where $\epsilon_{mj}(p_{mj}) = \frac{-f'_{mj}(p_{mj})p_{mj}}{f_{mj}(p_{mj})}$, the point price elasticity of the attraction function for channel m .

Given that the online price is a constant across locations, one can rewrite Eq. (6.5) to conclude the following: the online margin is a weighted linear combination of brick margins and certain constant terms across all locations. Note that the weights represent the normalized online demand by location, which in turn, are a function of the prices but are numbers less than 1. Here, the constant term represents the difference in the price over the point elasticities between the online and the brick channel in each location. The result is an interesting insight that can be shared with the retailer in order to better understand how optimal brick and the online channel prices interact with each other. Analogous to corollary 1, we can deduce the following result for the multi-location setting.

COROLLARY 2. Let $w_j(\mathbf{p})$ be the normalized values corresponding to the online demand in each location, i.e., $\tau_j \frac{f'_{oj}(p_o)}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})}$, over all the locations $j \in J$ resulting in $\sum_{j \in J} w_j(\mathbf{p}) = 1$. Then, for an MNL demand model where $f_{mj}(p_{mj}) = e^{a_{mj} - b_{mj}p_{mj}}$, Eq. (6.5) reduces to:

$$p_o^* = c_o + \sum_{j \in J} w_j(\mathbf{p}) \left[p_{bj}^* - c_{bj} + \frac{1}{b_{oj}} - \frac{1}{b_{bj}} \right] \quad \forall o, b \in M, \quad (6.6)$$

whilst for a linear attraction demand model where $f_{mj}(p_{mj}) = a_{mj} - b_{mj}p_{mj}$, Eq. (6.5) reduces to:

$$p_o^* = \frac{c_o}{2} + \sum_{j \in J} w_j(\mathbf{p}) \left[p_{bj}^* - \frac{c_{bj}}{2} + \frac{a_{oj}}{2b_{oj}} - \frac{a_{bj}}{2b_{bj}} \right] \quad \forall o, b \in M. \quad (6.7)$$

On the other hand, for a power attraction demand model where $f_{mj}(p_{mj}) = a_{mj}p_{mj}^{-b_{mj}}$ Eq. (6.5) reduces to

$$p_o^* = \left(c_o + \sum_{j \in J} w_j(\mathbf{p}) \left[p_{bj}^* \left(1 - \frac{1}{b_{bj}} \right) - c_{bj} \right] \right) \left[\sum_{j \in J} w_j(\mathbf{p}) \left(1 - \frac{1}{b_{oj}} \right) \right]^{-1} \quad \forall o, b \in M. \quad (6.8)$$

The above constraints are similar to the single location constraint except that the location specific terms (or their combinations) are replaced by the weighted averages. Therefore, the insights developed in the single location setting (which channel price is lower and by how much) hold here but with respect to the weighted average values. The insight that can be derived is if the brick price coefficient at every location is smaller in magnitude than the corresponding online coefficient then the optimal online prices will be lower than the brick prices assuming also that the unit cost online is less than its brick counterpart.

7. Exact methods to solve the OCP problem

In the following section, we provide two tractable methods that can achieve the global optimum to the OCP problem and compare their computational performance. These methods exploit the special structure of the OCP problem, the features of the attraction demand model, and the discreteness of the price space.

7.1. A decomposition method for the case of the brick and online channel

In the two channel case, a decomposition algorithm for the OCP problem with any omni-channel demand model is as follows: *fix the online price and solve the corresponding single channel brick problem to optimality, and repeat this search over all online prices.*

Given an online price, the problems at each of the brick locations are separable except in the presence of inter-location constraints, if any. In the absence of inter-location constraints, these brick problems are solved separately to optimality. For attraction demand models, we derive near-closed form solutions to the brick problems, enabling us to solve OCP fast enough for (near) real-time price optimization. Towards the end of the section, we provide extensions of this decomposition method in the presence of inter-location constraints. The method can also be generalized and remains viable for demand functions that do not necessarily come from the attraction family.

Consider the OCP problem employing an attraction demand model without inter-location constraints. Recall that channel and location specific restrictions are considered as inter-channel constraints. Given an online price p_o , the OCP problem decomposes by brick locations. If $\Pi(p_o)$ denotes the optimal objective of the corresponding OCP with online price p_o then

$$\Pi(p_o) = \sum_{j \in J} \tau_j \Pi_j(p_o) \quad (7.1)$$

where $\Pi_j(p_o)$ is the optimal objective of the corresponding OCP at location j . We denote this problem as $\text{OCP}_j(p_o)$. The only decision variable in this location specific OCP is the brick price denoted by p_{bj} . Therefore, given p_o , all the inter-channel constraints are reduced to lower and upper bounds on the scalar variable p_{bj} denoted by $\underline{h}(p_o)$ and $\bar{h}(p_o)$ respectively. The problem $\text{OCP}_j(p_o)$ takes the following form:

$$\text{OCP}_j(p_o) : \quad \Pi_j(p_o) = \max_{p_{bj} \in \Omega_{bj}} \sum_{m \in \{o,b\}} (p_{mj} - c_{mj}) \frac{f_{mj}(p_{mj})}{1 + f_{bj}(p_{bj}) + f_{oj}(p_o)} \quad (7.2)$$

$$s.t. \quad \underline{h}_j(p_o) \leq p_{bj} \leq \bar{h}_j(p_o) \quad (7.3)$$

We show below that $\Pi_j(p_o)$ can be obtained in *near* closed form using the following theorem. In the remainder of the section, we work under [assumptions 1](#) and [2](#).

THEOREM 1. *The optimal solution and the objective of the location specific OCP with inter-channel constraints and continuous brick prices (i.e., $\Omega_{bj} = \mathbb{R}$), given the online price p_o , for any location $j \in J$ can be simplified as follows:*

$$p_{bj}^*(p_o) = \begin{cases} \underline{h}_j(p_o), & \text{if } \hat{p}_{bj} < \underline{h}_j(p_o), \\ \hat{p}_{bj}, & \text{if } \underline{h}_j(p_o) \leq \hat{p}_{bj} \leq \bar{h}_j(p_o), \\ \bar{h}_j(p_o), & \text{otherwise.} \end{cases} \quad (7.4)$$

$$\Pi_j(p_o) = \begin{cases} Z_j(p_o, \underline{h}_j(p_o)), & \text{if } \hat{p}_{bj} < \underline{h}_j(p_o) \\ (p_o - c_o) \frac{f_{oj}(p_o)}{1 + f_{oj}(p_o)} + H_j(p_o, \hat{p}_{bj}), & \text{if } \underline{h}_j(p_o) \leq \hat{p}_{bj} \leq \bar{h}_j(p_o), \\ Z_j(p_o, \bar{h}_j(p_o)), & \text{otherwise.} \end{cases} \quad (7.5)$$

where

- $Z_j(p_o, p_{bj})$ is the objective of $OCP_j(p_o)$ with online price p_o and brick price p_{bj} ,
- $\hat{p}_{bj} = g_{bj}(z)$, $g_{bj}(\cdot) = f_{bj}^{-1}(\cdot)$,
- $H_j(p_o, \hat{p}_{bj}) = -\frac{z^2}{1 + f_{oj}(p_o)} g'_{bj}(z) = -\frac{f_{bj}^2(\hat{p}_{bj})}{1 + f_{oj}(p_o)} \frac{1}{f'_{bj}(\hat{p}_{bj})}$ and
- z is the solution of the differential equation:

$$g_{bj}(z) + z \left(1 + \frac{z}{1 + f_{oj}(p_o)} \right) g'_{bj}(z) \stackrel{\text{def}}{=} \sigma_j(p_o) = (p_o - c_o) \frac{f_{oj}(p_o)}{1 + f_{oj}(p_o)} + c_{bj}. \quad (7.6)$$

In particular, for an MNL demand function, where $f_{mj}(p_{mj}) = e^{a_{mj} - b_{mj} p_{mj}}$ for $m \in \{o, b\}$, the differential equation simplifies to

$$\log x + x = a_{bj} - 1 - b_{bj} \sigma_j(p_o) - \log(1 + f_{oj}(p_o)), \quad (7.7)$$

where $z = x(1 + f_{oj}(p_o))$ and $H_j(p_o, \hat{p}_{bj}) = \frac{x}{b_{bj}}$.

The proof of the theorem is in [Appendix A](#). The unconstrained objective at optimality, $\Pi_j(p_o)$, has an interesting interpretation:

$$\underbrace{(p_o - c_o) \frac{f_{oj}(p_o)}{1 + f_{oj}(p_o)}}_{\text{Profit in the absence of the brick channel}} + \underbrace{H_j(p_o, \hat{p}_{bj})}_{\text{Incremental profit due to the brick channel}}.$$

The first term is the profit from the online channel in location j when the brick price in location j is set to a very large number so that, for all practical purposes, the brick channel does not exist. The second term is the additional profit from location j when brick price is set to its optimal value \hat{p}_{bj} . Note that this additional profit is always non-negative because $g'_{bj}(z) = [f'_{bj}(\hat{p}_{bj})]^{-1} \leq 0$ because of [assumption 1](#).

Let $\psi_j(z, p_o)$ denote the left hand side (LHS) of [Eq. \(7.6\)](#).

PROPOSITION 3. $\psi_j(z, p_o)$ is a non-increasing function in z and in p_o . Moreover, for a given p_o , $\lim_{z \rightarrow 0} \psi_j(z, p_o) = \infty$, and $\lim_{z \rightarrow \infty} \psi_j(z, p_o) \leq 0$ while $\sigma_j(p_o)$ is a constant independent of z . This implies that Eq. (7.6) always has a solution.

The solution to Eq. (7.6) can be derived using root finding algorithms such as the Newton Raphson method.

The above theorem and proposition assumes continuous brick prices. The extension to the case of discrete brick prices is simple. Each subproblem by location given p_o is a concave maximization problem in a market share space. Therefore, a simple rounding algorithm around the maximum value \hat{p}_{b_j} that checks for the maximum objective at the ceiling and floor of \hat{p}_{b_j} with respect to the discretization within the feasible region can be employed.

In order to find the optimal online price, $p_o^* = \operatorname{argmax}_{p_o} \Pi(p_o) = \operatorname{argmax}_{p_o} \sum_{j \in J} \tau_j \Pi_j(p_o)$, that solves the OCP, we search over every discretized online price point with the near-closed form solution described above for each location $j \in J$. Note that the example in Fig. 3 depicts a function that is non-convex and can have multiple peaks. This is the reason that the objective has to be evaluated at every feasible online price point.

REMARK 6. *The complexity of the decomposition algorithm in the absence of inter-location constraints and an attraction demand model is $O(IJR)$ where I is the size of the price ladder in the online channel, J is the number of locations, and R is the complexity of a root finding algorithm. In other words, the computational complexity of the decomposition algorithm is pseudo-linear.*

The analysis so far ignored inter-location constraints. If inter-location constraints are active, the following approach can be adopted in practice. For a chosen online price, all feasible brick prices, from a discrete set, for each location corresponding to the online price and its corresponding contributions to inter-location constraints and objective are included in a master problem. The master problem maximizes the total profit subject to the inter-locations constraints only. The feasible brick prices in each location correspond to binary variables in the master problem and one feasible brick price per location has to be chosen. This master problem has a structure identical to a multi-choice multi-dimension knapsack problem (multi-choice corresponds to picking one price in every location and multi-dimensions corresponds to the inter-location constraints). For example, in the presence of a single inter-location constraint such a global volume goal, we obtain a multi-choice knapsack problem (MCKNP), which can be solved using pseudo-polynomial algorithms based on dynamic programming (Pisinger 1994). One such MCKNP has to be solved for every chosen online price. Therefore, the overall method for the OCP problem remains pseudo-polynomial.

The decomposition method described in this section are especially beneficial when there are several inter-channel constraints (e.g., channel price monotonicity) because they can all be managed locally with the brick subproblem. The above knapsack-based extensions, generalize to omni-channel demand models beyond the attraction family.

7.2. A mixed-integer programming approach for the multi-channel case

We present a MIP re-formulation that simultaneously works in the price and the market share space in order to handle a variety of practical constraints. Introducing market share variables enables us to avoid bilinear binary variables (e.g., $z_{bli}z_{oli}$ for channels b at price \bar{p}_i and channel o at price \bar{p}_i in location l) in the formulation, leading to tractable runtimes for practical instances. This formulation is flexible in that it can encode important and complex business rules employed in practice, and can overcome the two-channel limitation of the decomposition method. On the other hand, the decomposition method is generalizable to demand functions beyond the attraction family, whereas the MIP approach gainfully exploits the structure of the attraction demand model in order to generate tractable market share transformations.

Our re-formulation is in the spirit of [Subramanian and Sherali \(2010\)](#) who developed a reformulation for price optimization problem using a hybrid MNL demand model by employing a piecewise linear approximation of the market size term. In contrast, our reformulation is valid for a general attraction demand model and does not require such piecewise linear approximations or big- M upper-bounds.

Let the feasible discrete prices for each channel $m \in M$ and location $j \in J$ be denoted by \bar{p}_{mji} for $i \in I_{mj}$. Here, the set I_{mj} denotes the index set of feasible prices. Let z_{mji} be a binary variable which is nonzero only if the price in channel $m \in M$ at location $j \in J$ is \bar{p}_{mji} . Note that for a virtual channel $v \in V$ the prices across all locations are the same. Therefore, the corresponding prices \bar{p}_{vi} , the price index set I_v and binary variable z_{vi} are location independent. For ease of exposition, we use the notation z_{vji} and assume that it is always replaced by variable z_{vi} for every $v \in V$. Similarly, we assume $I_v = I_{vj}$ and $\bar{p}_{vi} = \bar{p}_{vji}$ for every $v \in V$. Assuming $q_{mji} = \tau_j(\bar{p}_{mji} - c_{mj})f_{mj}(\bar{p}_{mji})$, $r_{mji} = f_{mj}(\bar{p}_{mji})$, $\alpha_{kmji} = A_{kmj}r_{mji}$ and $\beta_{kmji} = B_{lmj}\bar{p}_{mji}$, the omni-channel price optimization problem is reduced to the following form:

$$\max_{z_{mji}, z_{vi}} \sum_{j \in J} \frac{\sum_{m \in M} \sum_{i \in I_{mj}} q_{mji} z_{mji}}{1 + \sum_{m \in M} \sum_{i \in I_{mj}} r_{mji} z_{mji}} \quad (7.8)$$

$$\sum_{j \in J} \sum_{m \in M} \sum_{i \in I_{mj}} \frac{\alpha_{kmji} z_{mji}}{1 + \sum_{m \in M} \sum_{i \in I_{mj}} r_{mji} z_{mji}} \leq u_k \quad \forall k \in K \quad (7.9)$$

$$\sum_{j \in J} \sum_{m \in M} \sum_{i \in I_{mj}} \beta_{lmji} z_{mji} \leq v_l \quad \forall l \in L \quad (7.10)$$

$$\sum_{i \in I_{mj}} z_{mji} = 1 \quad \forall m \in M, j \in J \quad (7.11)$$

$$z_{vi} = z_{vji} \quad \forall i \in I_v, v \in V, j \in J \quad (7.12)$$

$$z_{mji} \in \{0, 1\} \quad \forall i \in I_{mj}, m \in M, j \in J \quad (7.13)$$

In the above formulation, the general business rules on volumes and prices given by constraints (5.2–5.3) are encapsulated in constraints (7.9–7.10). In practice, the z_{vji} variables are never introduced and we work only with z_{vi} variables, and constraint (7.12) is redundant. The above formulation is nonlinear, but can be linearized as follows. We first use the fractional programming transformations proposed by Charnes and Cooper (1962) to overcome the nonlinearity arising from the ratio terms, and then use the reformulation and linearization technique (RLT) proposed by Sherali and Adams (1999) to eliminate the non-linearities due to product terms. The RLT transformations exploit the discrete nature of the binary variables, allowing us to recover an exact reformulation of the OCP problem. We now describe the transformations. Let

$$y_j = \frac{1}{1 + \sum_{m \in M} \sum_{i \in I_{mj}} r_{mji} z_{mji}} \quad \forall j \in J. \quad (7.14)$$

Because z_{mji} are binary variables and r_{mji} are non-negative constants, $0 \leq y_j \leq 1 \forall j \in J$. Now define $x_{mji} = y_j z_{mji}$. It is easy to see that $0 \leq x_{mji} \leq 1 \forall m \in M, i \in I_{mj}$. Substituting these transformations and linearizing, the resulting reformulated OCP problem is as follows:

$$\max_{z_{bj}, z_{oi}, y_j, x_{mji}} \sum_{j \in J} \sum_{m \in M} \sum_{i \in I_{mj}} q_{mji} x_{mji} \quad (7.15)$$

$$\sum_{j \in J} \sum_{m \in M} \sum_{i \in I_{mj}} \alpha_{kmji} x_{mji} \leq u_k \quad \forall k \in K \quad (7.16)$$

$$\sum_{j \in J} \sum_{m \in M} \sum_{i \in I_{mj}} \beta_{lmji} z_{mji} \leq v_l \quad \forall l \in L \quad (7.17)$$

$$y_j + \sum_{m \in M} \sum_{i \in I_{mj}} r_{mji} x_{mji} = 1 \quad \forall j \in J \quad (7.18)$$

$$x_{mji} \leq y_j \quad \forall i \in I_{mj}, m \in M, j \in J \quad (7.19)$$

$$x_{mji} \leq z_{mji} \quad \forall i \in I_{mj}, m \in M, j \in J \quad (7.20)$$

$$\sum_{i \in I_{mj}} x_{mji} = y_j \quad \forall m \in M, j \in J \quad (7.21)$$

$$\sum_{i \in I_{mj}} z_{mji} = 1 \quad \forall j \in J, m \in M \quad (7.22)$$

$$z_{vi} = z_{vji} \quad \forall j \in J, v \in V \quad (7.23)$$

$$z_{mji} \in \{0, 1\} \quad \forall i \in I_{mj}, m \in M, j \in J \quad (7.24)$$

$$y_j, x_{mji} \geq 0 \quad \forall i \in I_{mj}, m \in M, j \in J \quad (7.25)$$

Constraint (7.18) linearizes Eq. (7.14). Constraints (7.19–7.20) along with the objective linearize the product term $x_{mji} = y_j z_{mji}$. RLT constraints (7.21) are implied by constraint (7.19) in the integer sense but serve to tighten the underlying LP relaxation. In our numerical computations, we observed that the addition of constraints (7.21) yielded a considerable improvement in the computational performance.

Observe that the above formulation is now a linear MIP and a commercial optimization software package like IBM ILOG CPLEX can be used to solve this problem to optimality. Furthermore, this transformed OCP formulation allows for any number of channels and can incorporate a variety of business rules that are commonly employed in practice.

7.2.1. Comparison of computational performance between the decomposition and the MIP approach Fig. 4 compares the average running time of the decomposition method and the MIP method for two channels (brick and online) as a function of the number of locations using simulated demand models that were motivated from real data. We assume the absence of inter-location constraints, and report run times for the decomposition methods under the assumption of full parallelization of subproblem solutions across locations. An MNL attraction model was used in the simulations, and the resulting optimization problem was solved using the decomposition method by exploiting the near-closed form results of Theorem 1. The simulations incorporated the business rules related to the price bounds and discrete prices. Not surprisingly, Fig. 4 shows that the *parallelized* decomposition method takes a constant time even with a large number of locations. On the other hand, the run times associated with the MIP method tends to increase with the number of locations. However, for a relatively small number of locations, the MIP approach was faster than the decomposition method. In practice, we employed no more than 100 locations/zones to model retail chains having thousand or more physical stores. The figure also includes the average run time of the MIP for a single online channel. It has a similar trend as that exhibited by the MIP method with two channels, but tends to grow at a slower rate as the number of locations increase.

8. OCP implementation for a major US retailer

In this section we report results of an OCP implementation for a major U.S. omni-channel retailer. We worked with our client and engaged with the retailer over the course of 8 months to demonstrate the business value of the integrated omni-channel regular pricing over their existing channel independent regular pricing method. We describe (a) the retailer and their business process (while maintaining their anonymity), (b) the business problem that motivates the new solution, (c) the data summary, (d) our implementation details and finally, (d) our business value assessment

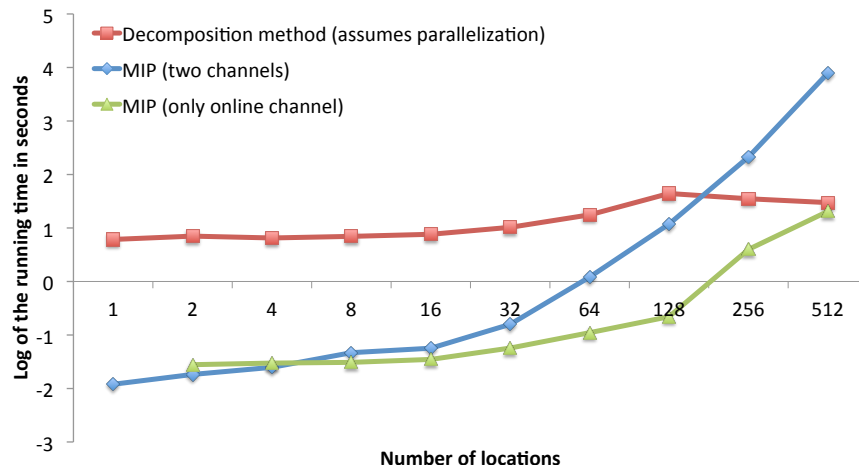


Figure 4 Average run times of the decomposition model and the MIP over 25 simulated instances using a MATLAB prototype as a function of the number of locations

for two product categories. The details about the commercial viability of the proposed solution, and integration into the client IT architecture is presented in [Section 9](#).

Retailer and their current business process: The omni-channel retailer we worked with sells a variety of products, including office supply product categories such as inkjet cartridges, markers and highlighters, cut-sheet paper and filing. They operate a brick-and-mortar channel with a network of well over 1500 stores across the United States. The online channel is used to complete sales transactions that are routed through their website, as well as mobile and paper-catalog orders.

The organizational structure of the retailer results in two different divisions managing the planning and operations of the brick-and-mortar, and the online channels and they are largely independent of each other. Both divisions use a regular price optimization (RPO) solution to manage prices for many non-perishable products, referred to as UPCs (universal product code) in the channels that they are responsible for. The prices for the remaining products are controlled partly by the manufacturer, or are price-matched with certain competitors. The incumbent RPO solution produces demand forecasts that are independent of the other channel or competitors, and identifies regular or base price for the (non-perishable) products that maximizes the retailer’s profitability over a specified finite horizon subject to some business constraints. One price is found for every geographical cluster of brick stores identified by the retailer as a ‘price zone’. The entire online channel is treated as a separate price zone. The regular prices identified are treated as ticket prices. Sometimes, the ticket prices are overlaid with promotions. Promotions can be of various types, and includes discounting and advertisement. The regular prices can be held constant for a few weeks, or can vary from week to week and the retailer can re-optimize it using the RPO solution

as needed. The pricing solution requires weekly sales and promotion data but does not have, nor require visibility into the inventory levels on the shelf. Standard inventory re-order policies are in place, and are managed by separate systems.

Business Problem: Among the UPCs that are priced by the retailer, more than 20% of the UPCs are sold in both channels - brick and online. They contribute to a significant portion of the retailer’s category revenue (details provided below). Due to the rapid growth of the online channel, the retailer was concerned about how to optimally coordinate prices between these two channels while accounting for competitor effects. In our discussion, the retailer did point that in the future, they would use the OCP solution to re-optimize online prices as often as possible, in order to respond to changes in competitor prices. Their immediate goal was to solve the business problem of coordinating regular prices between channels in order to optimize certain Key Performance Indicators (KPIs) such as gross profit and sales volume. The retailer also mentioned that they were not keen on enforcing a price match between the brick and online channels.

Data Summary: To support the business value assessment, we were provided with 52 weeks of U.S. sales transaction and promotion data (date range of July 2012 to July 2013) for the brick and online channel for two categories: (1) inkjet cartridges and (2) markers and highlighters. The top 50 UPCs in terms of historical volume that were sold in both the online and the brick and mortar channel (channel volume share of at least 1%) were selected for the business value assessment. Some statistics about the data are summarized in [Table 8](#).

Category	No. of UPCs	Avg. Final Price		% of category revenue	Online volume share
		Brick	Online		
Inkjet Cartridges	50	\$36.4	\$32.1	30%	12%
Markers and Highlighters	50	\$8.7	\$8.3	42%	12%

Table 1 Summary of the 2012-13 data provided.

In the inkjet cartridges and the markers and highlighters category, the 50 UPCs that were selected contributed to about 30%, and 42% of the category revenues respectively, and have a 12% online volume share in each case. Although the online share of the retailer sales in 2012-13 was relatively low, this number has been steadily increasing year to year.

We also note that the inkjet cartridges category consists of products that are more expensive than the markers and highlighters category and that the retailer sells products in the brick channel at a slightly higher price than the online channel.

There were approximately 40 distinct geographical price zones in the brick channel. The continuum of online sales was disaggregated by the brick channel’s geographical price zones using the

omni-channel framework described in [Section 3](#). The total sales rates across the 40 zones were not evenly distributed across locations, and we found that the top 10 zones contributed to 54% of the total sales, and the top 20 zones to about 83% of the total sales.

For each of the UPC-zone pairs, we obtained by channel, the weekly aggregated sales, volume weighted weekly average ticket price, discounts and promotions, weekly holidays and seasonalities. Seasonality is a category level time series computed from multi-year data provided as a part of input data from an upstream existing data processing system. For a small sample of UPCs, we were also provided a time series of three online competitor’s final website prices. We observed that the products in both these categories exhibited a relatively steady sales rate, which is typical of non-perishable basic products.

We were also provided with the wholesale cost information for the different UPCs and this was the same across the sales channels. It is possible that the holding cost is lower online but is offset by the shipping cost, although this information was not provided to us.

In the following subsections, we (a) estimate and compare the degree of channel-switching in each of the two categories using historical data, (b) calculate the cross-channel price elasticities at the UPC-zone level using the proposed omni-channel demand model [Eq. \(4.2\)](#) and (c) report on the results of the business value assessment.

8.1. Identifying UPCs having significant cross-channel price elasticity

In this section we highlight an empirical threshold based method that we developed for our client to help them identify UPCs exhibiting significant cross-channel price elasticity effects. These UPCs tend to be ideal candidates for omni-channel price optimization.

Because the assumption of substitution is implicit within an attraction demand model form, we employed an alternative scan-pro demand model that does not assume substitution or complementary effects between channels. We fit a scan-pro log-linear demand model to the historical sales data at the UPC-zone-channel level using own-channel and cross-channel features as follows:

$$\begin{aligned} \log(\text{Channel Sales}) = & \alpha_0 + \sum_{m \in M} \left[\alpha_{1,m} \text{PRICE}_m + \sum_k \alpha_{2,k,m} \text{PROMOTION-VARIABLES}_{k,m} \right] \quad (8.1) \\ & + \sum_{m \in M, j} \alpha_{3,j,m} \text{COMPETITOR PRICES (optional)}_m + \sum_k \alpha_{4,l} \text{TEMPORAL-VARIABLES}_l \end{aligned}$$

where $m \in M$ represent the different channels such as brick and online. The own and cross-channel features include factors such as price and promotion variables. The latter can include discounts and other indicators represented by index k . The causals also include the temporal variables such as the seasonality, holiday effects and trends denoted with index l , and competitor prices, whenever

present. Note the difference of this model from a single channel model where all the causals depend only on channel m .

Using a standard glmnet R statistical package (Friedman et al. 2010), we performed a constrained linear regression with lasso regularization, where we enforce appropriate sign constraints consistent with cross-channel substitution (e.g., non-positivity of own price coefficient, as well as the non-negativity of the cross-channel price coefficients to be estimated). These constraints are added in order to eliminate non-intuitive and incidental complementary cross-channel effects, which, if not removed, could produce false-positive results that exaggerate the cross-channel impact of features. We also used an appropriate lasso (L1) penalty setting to ensure that only significant features are selected. Using the results of this estimation, we can classify a UPC-zone-channel instance as having a cross-channel price elasticity if the feature is selected within the calibrated log-linear model. The magnitude of the impact viewed as cross-channel elasticities is provided in the following section.

Category	Percentage of UPCs that have significant cross channel effects
Inkjet Cartridges	76%
Markers and Highlighters	56%

Table 2 Presence of significant cross-channel effects

In [Table 2](#) we present the percentage of UPCs in each category having significant cross-channel price effects, which is in turn, defined as having cross channel effects in at least 10% (four) of the zones. This turns out to be 76%, and 56% for the inkjet cartridges and the markers and highlighters categories, respectively. These results were presented to the retailer, who concurred with the findings based on their own experience, indicating that the channel switching based on pricing does occur, particularly within the more expensive inkjet cartridges category. Thus, we are already observing cross-channel demand influence in the 2012-13 data, when the online share was no more than 12%.

If a UPC-zone-channel exhibits cross-channel price effects, then that zone’s brick channel has to be jointly optimized for price with all the zones of the online channel, which in turn, also have to be optimized together because of the presence of the online price constraint linking all the virtual zones. On the other hand, if the cross-channel price effect is negligible in either direction, then that specific UPC-zone’s brick channel can be priced independently in the absence of inter-channel and inter-location constraints.

Although a calibrated scan-pro demand model can potentially be used for price optimization, our preferred choice is an attraction demand model due to multiple practical considerations as discussed

in [Section 4](#). For example, a scan-pro model in this setting requires twice the number of parameters to be estimated as the attraction demand model. Furthermore, the resultant optimization problem using a MIP approach requires the creation of second order binary variables that can result in intractable run times. Therefore, the scan-pro model helps in identifying UPCs/categories that are ideal candidates for omni-channel price optimization. This feature, in itself, was a practical and useful tool for the client and in turn, the retailer.

8.2. Estimating cross-channel elasticities

In this section, we use a omni-channel demand model to calculate cross-channel price elasticities. We construct a demand model for each UPC-zone in order to account for heterogeneity in channel preferences across the geography. Within a UPC-zone, a customer chooses between no-buy, buy in the online channel or buy in the brick-and-mortar channel. We assume that the demand model has a form described in [Eq. \(4.2\)](#) based on the MNL attraction function.

Model selection and cross-validation on a variety of training instances yielded the following log-utility function for a given channel:

$$\begin{aligned} \log(\text{Channel Attraction}) = & \beta_0 + \beta_1 \text{PRICE} + \sum_k \beta_{2,k} \text{PROMOTION-VARIABLES}_k \quad (8.2) \\ & + \sum_j \beta_{3,j} \text{COMPETITOR PRICES (optional)}. \end{aligned}$$

The best short-term forecast for market size for a zone (or zone-channel for single channel demand model) was given by:

$$\log(\text{Market Size}) = \gamma_0 + \sum_{1,k} \gamma_{1,k} \text{TEMPORAL-VARIABLES}_k. \quad (8.3)$$

The promotion variables included discounts and other promotional indicators and the temporal variables included seasonality, holiday effects and trend. Competitor prices were introduced as attributes in the channel specific utilities, whenever they were available. For certain retailer data sets, we observed that it was important to incorporate the temporal effects in the channel attraction model instead, because the lead time required to complete online order delivery can result in holiday orders being placed earlier in the online channel, thereby shifting the time-period of peak sales in different channels. This demand model was calibrated using the approach discussed in [Section 4](#).

The preferred method for the retailer to track the forecast accuracy was using the weighted mean absolute percentage error (WMAPE) metric where t represents the week index:

$$WMAPE = \frac{\sum_t |\text{predicted sales}(t) - \text{actual sales}(t)|}{\sum_t \text{actual sales}(t)} * 100 \quad (8.4)$$

Table 3 reports the achieved out-of sample WMAPE metric for eight-week ahead predictions of weekly sales at the UPC-zone level. On the whole, the achieved forecast accuracy at this fine level of aggregation satisfied the retailer’s expectation.

Category	Brick	Online
Inkjet Cartridges	29	36
Markers & Highlighters	36	24

Table 3 Eight week average out-of-sample WMAPE fit over the UPC-zone pairs in each category.

Own and cross-channel price elasticities: Price elasticity is defined as the percentage change in demand in a specific channel that is associated with a percentage change in a price variable. If the price change belongs to the same channel where the sales change is measured, we refer to it as own price elasticity, and if it corresponds to the other channel, we refer to it as a cross-channel price elasticity. Price elasticity is dimensionless, and tends to have a significant impact on the optimized prices. It is commonly studied and viewed as a property of a product. We refer the reader to Train (2009) for calculating the own and cross-elasticity for an MNL demand model.

Table 4 summarizes the average and the range of the own and cross-channel price elasticities evaluated at the average channel price. These elasticities range between -2.0 to 0. The relatively low elasticity values are typical of non-perishable consumer products that represent obligatory transactions rather than luxury purchases. It can also be seen that the cross-channel price elasticities are significant for these categories, although their impact is expectedly lower than that of the own channel price. From Table 4, we observe that cross-channel price elasticities can be as high as 50% of the own channel price elasticity. Note that these cross-elasticities have a positive sign which is indicative of substitution between channels, i.e., channel-switching behavior based on the price differential. Note also that the cross-elasticities are asymmetric in that the impact of brick prices on the online sales is different from (and tends to be higher than) the impact of the online prices on brick sales. It is indicative of the heterogeneity of the customers shopping in the different channels as well as the volume share of these channels (the absolute change in volume of brick sales is much higher than that for the online channel). As the online share rises in the future, we can expect the online price to exert an increasing influence on the brick channel sales.

It must be noted that the cross-channel elasticities (off diagonal entries in Table 4) cannot be computed using traditional pricing models employed in the industry. Specifically, the omni-channel framework proposed in this paper enables us to compute the impact of the brick prices on online sales at a location specific level (lower off-diagonal entry). Similarly, we compute the elasticities of online competitor prices, whenever available. The pricing data for brick competitors was sparse

Channel \ Price	Inkjet Cartridges		Markers and highlighters	
	Brick price	Online price	Brick price	Online price
Brick sales	-0.66 (-1.84,-0.4)	0.06 (0,0.22)	-1.19 (-1.87,-0.01)	0.04 (0, 0.05)
Online sales	0.31 (0,1.46)	-1.04 (-1.99,-0.01)	0.19 (0,0.39)	-0.78 (-1.96,-0.01)

Table 4 Average and the range (10th and 90th quantile) of the own and cross-channel price elasticities.

and hence could not be included. Complete brick competitor data, if available, can be similarly analyzed to determine the impact of a competitor retail chain’s brick stores on online sales.

The calibrated zone level omni-channel MNL demand models were used to perform the OCP optimization for the two product categories and we report on the business value in the following section.

8.3. Business value assessment

As a part of the OCP implementation, we worked with the retailer and our client to set up a business value assessment. Toward this, we held a series of conversations to understand their omni-channel business process, channel sales goals, pricing policy, and the competitive environment they operated in. We learned that:

1. Online prices tended to lower than the brick store prices.
2. Their brick stores were able to retain an 88% channel share, suggesting that the vast majority of their customers preferred to touch-and-feel, and purchase the product in-store rather than wait, unless of course, a substantial price bargain was available online.
3. Maintaining and then growing their online presence was important in order to remain competitive, given the steadily rising popularity of the online shopping. The lower online prices appeared to be partly driven by the need to compete with large online retailers who offered the same product at an even lower price.
4. Raising prices in all channels may yield short-term profits, but will result in lost customers in the long run. In fact, their incumbent business process that optimized channel prices independently was more likely to produce such a result for basic products that are weakly price elastic.

Given these findings, the challenge for us was to solve a multiobjective problem that achieves an optimal balance between brick and online prices for the next few weeks in order to be more profitable, while also preserving global as well as online sales volume. To demonstrate the business value of our solution, we analyzed the top 50 UPCs in the two product categories and jointly optimized UPC prices in all the brick and mortar stores and the online channel using the calibrated omni-channel MNL demand model as described in the previous subsection. We incorporated price

bounds to ensure that recommended prices were within historical values, and satisfy the magic number endings. To preserve sales volume, we added a global volume goal that required the predicted volume at the optimal prices to be no less than the predicted volume at the baseline (actual) prices. To prevent the myopic response of raising prices in both channels, we added a price balancing constraint that required the average channel price to be no higher than the corresponding baseline value. For the business case, we explicitly focused on the multi-period regular pricing problem wherein the goal is to find the optimized price for a product over a specified time horizon which was chosen as the last 8 consecutive out-of-sample prediction weeks in 2013. The resultant optimization models developed as a JAVA API were evaluated on a Windows-7 PC having 8GB RAM, and an Intel Core i7 processor. CPLEX 12.6.2 with its out-of-box parameter settings was used to solve the resultant MIPs to global optimality. The average runtime per UPC was 1.7 seconds, and no more than 3 seconds in the worst case. Thus, the solution response is fast enough to process multiple price updates within a day, e.g., within a dynamic price optimization scenario in a production environment.

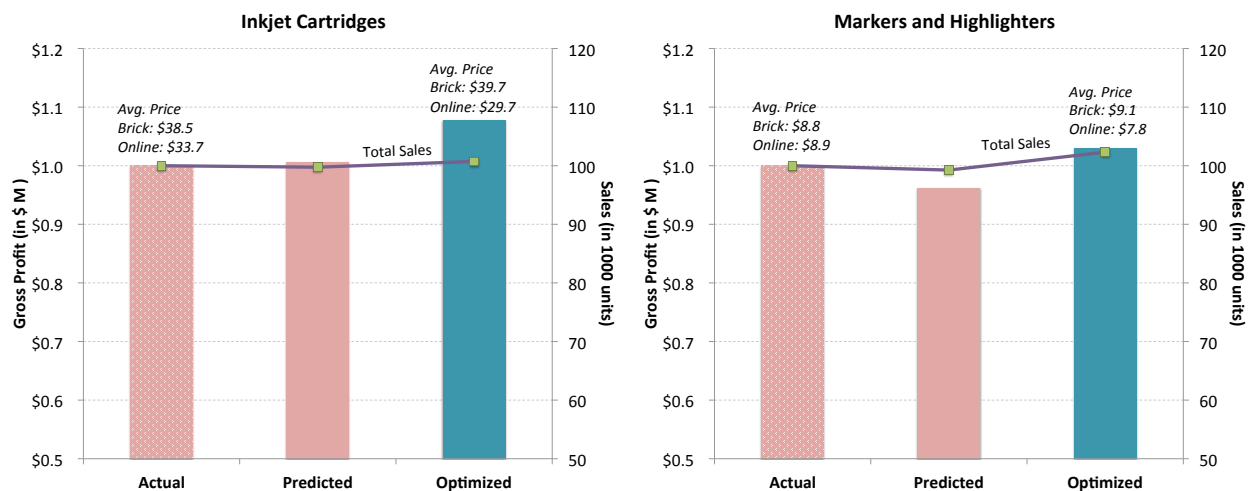


Figure 5 Re-normalized gross-profit and sales during a 8 week period for actuals from historical data, predicted using omni-channel demand models at actual prices and the optimization

The results of this optimization are presented in Fig. 5 along with two baselines. The first baseline (actual) represents the KPIs achieved by the retailer’s incumbent single channel forecasting and pricing system. The second baseline (predicted) represents the KPIs using the actual prices and the omni-channel demand model. To protect the retailer’s data privacy, the actual gross profits are normalized to \$1M and the sales volume to 100K units and hence the results for predicted and optimized are relative to these normalized actuals. We can observe that the predicted and realized (or actual) metrics are relatively close to each other for both categories, i.e., within 1% in terms

of sales volume and gross profit each for inkjet cartridges category, and within 1% in terms of sales volume and 4% in terms of gross profit for markers and highlighters category. The optimized metrics, on the other hand, indicate a 7% gross profit lift in the inkjet cartridges and the markers and highlighters categories each with respect to the predicted metrics, while also achieving a 1% and 3% increase in aggregate sales volume respectively.

Fig. 6 displays the histogram of the relative change in brick store prices compared to the baseline. In the inkjet category, we observe that the optimization increased brick prices in 70% of the zones, while retaining or lowering prices in 30% of the locations. On the other hand, for the markers, only 37% of the brick locations witness a price rise (due to relatively higher own brick price elasticity). The optimal price gap between the channels can be controlled via another inter-channel price constraint, if required.

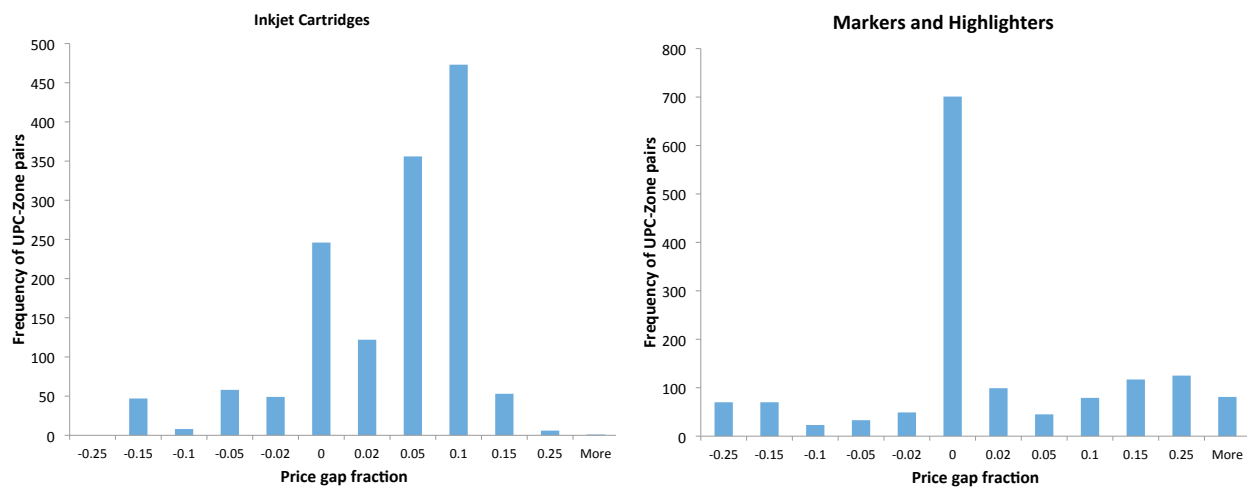


Figure 6 Histogram of relative price change in brick store prices compared to the actuals

To see how the OCP approach generates better KPIs for retailers, we first observe from the price histogram that the average optimized online price discounts the actual prices calculated by the existing single-channel system, while the optimized average brick store prices are marginally higher. Note that the retailers existing single-channel systems are limited by their silo approach. Such an uncoordinated approach can achieve incremental profitability only by raising the average price across all channels, at the expense of losing sales volume. Such a solution can be infeasible because it can fail to satisfy the global volume and balanced pricing constraints. The OCP approach overcomes this challenge as follows. First, it analyzes the feasible channel price combinations via integrated optimization. Second, it recognizes that a portion of brick customers who are unwilling to pay a higher brick price will switch channels if an attractive online discount is on offer, thereby recapturing a fraction of lost brick sales due to a price increase at any store location. By carefully

searching this feasible space of joint pricing decisions, the OCP approach finds the most profitable recommendation that satisfies the global constraint of preserving sales volume. As an additional benefit, such a pricing solution boosts the online sales, allowing the retailer to be more competitive with e-tailers in the marketplace.

Our business value assessment projected an incremental annual profit gain of 7% for the retailer for the categories analyzed. These results were presented at the retailer site to a team of pricing analysts and senior executives, including their Vice-President for revenue management. Their feedback was positive, and with similar experiences with other retailers in a variety of product categories, our proposed solution was approved for commercial deployment.

9. Deployment and Commercialization

Fig. 7 provides a high level view of the data flow of the OCP implementation. A big-data platform was employed to implement the specific data extraction algorithms required to create the omni-channel modeling framework. This platform is scalable to the enterprise level and allows for extraction, transformation, loading (ETL), and is capable of managing large volumes of diverse transaction data (e.g., in the order of several Terra Bytes) associated with various channels and locations, including competitor prices, local events information, product ratings, social sentiment, etc. A D3 data visualization tool was employed to view a variety of results derived from the omni-channel framework. The data preprocessing is followed by the omni-channel models that includes demand prediction and the optimization engine. A scalable JAVA API (application programming interface) employing CPLEX 12.6.2 was developed for commercial deployment. JAVA programming language and CPLEX were chosen to guarantee compatibility on a variety of operating systems. The API is configurable with a variety of business rules and goals that can vary across UPCs and allows for an in-memory processing of inputs and outputs.

The data flow described in Fig. 7 and the JAVA API was specifically designed to integrate with the existing IBM Commerce IT infrastructure. Through multiple engagements with retailers we were successful in fine-tuning the OCP solution so as to meet infrastructure and operational requirements. Proprietary versions of these models were developed and handed to IBM Commerce following multiple sessions of knowledge transfer. These assets were deployed into production by IBM Commerce in 2014 and reside in the IBM cloud. The methods and the systems described in this paper are patent pending.

The OCP solution emphasizes to retailers the importance of integrated pricing decisions across the retailer's sales channels and represents a major shift from their traditional approach of pricing in channel silos. Along with the double digit growth rate of online, this solution is proving invaluable to retailers, as it positions them strategically in this omni-channel era.

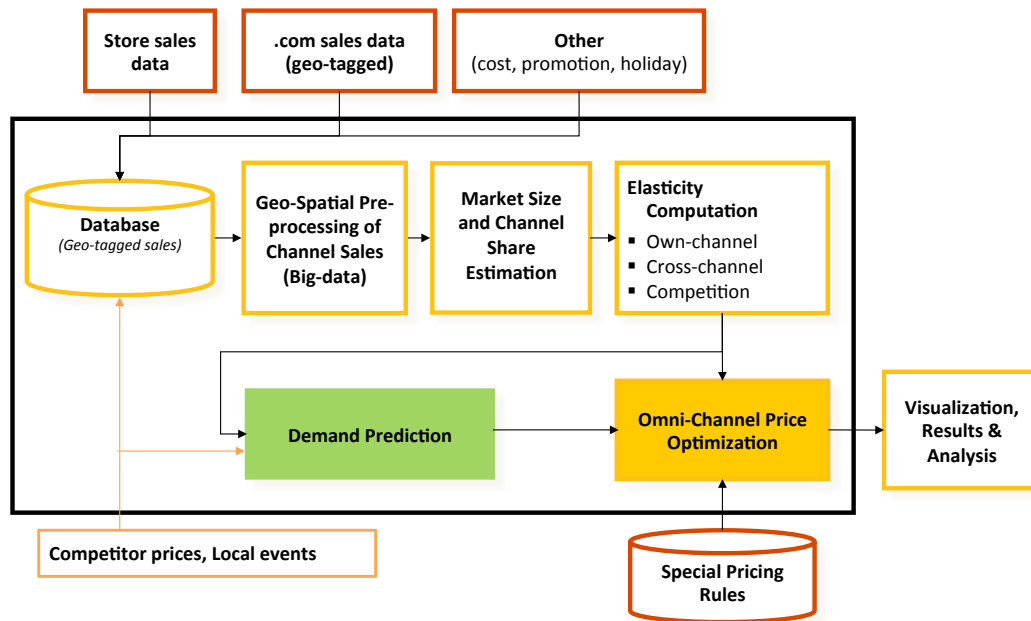


Figure 7 Data flow diagram for OCP

9.1. Impact of operations research in the deployment of the OCP models

We now delineate three applications of the OCP optimization model and discuss the positive impact of operations research on the resulting pricing decisions and prior business practices.

Value of accurate multi-period price optimization modeling: The pricing problems sometimes require a multi-period treatment even in the absence of inventory effects because the retailer aims to set a fixed regular price over a time horizon, and runs promotional campaigns to alter prices in near term. The business value assessment analyzed in this paper corresponds to this multi-period setting. Note that in these scenarios the channel demands are predicted for every period in the planning horizon using the weekly forecasting reports. A heuristic optimization method that was previously adopted was to solve the pricing problem for a representative ‘average’ time period that ignores the temporal variation in demand, resulting in suboptimal solutions. We demonstrated as part of the business value assessment that the OCP model can be easily extended to include multiple time periods and its demand variations, via a constraint similar to [constraint 5.4](#) that runs across time and solve the resulting problem with no significant increase in run time.

Profitability threshold based price matching and the practical value of global optimality: Consider a retailer who would rather price-match the channels if the profitability gain (by not imposing the constraint) is insufficient. The retailer can conveniently specify this tradeoff via a profitability threshold, where in, a price match constraint is accepted only if the resultant

drop in profitability is within the threshold limit. One can view this threshold policy as a means of understanding the implications of a strategic decision like price matching.

A natural way of implementing this feature is to solve the OCP problem with, and without the price-matching constraint and then choose the preferred solution. Employing an optimal solution approach to OCP turns out to be critical in this context. The use of a local-optimum based heuristic approach to solve the OCP problem with and without the price matching constraint, can result in incorrect profitability gaps, producing ‘false-positive’ price-matching recommendations. Note that for such heuristic methods, a price-matching constraint can operate like a cutting-plane that deletes a local (but not global) optimal solution to the unconstrained problem (i.e., without the price-matching constraint), potentially yielding an improved profitability objective. In such cases, the heuristic approach is likely to approve price-matching, whereas in reality, the achieved unconstrained profit value may have been far away from optimality. Numerical testing showed that such false-positives were not uncommon. Furthermore, when the user employs the application in an interactive ‘what-if’ mode to analyze the impact of omni-channel constraints, such behavior becomes quite apparent, and has a negative influence on user experience and acceptance. On the other hand, an optimal approach always generates the correct price-coordination recommendation, and the application produces stable and predictable responses from a user perspective.

Asynchronous channel-specific dynamic price optimization to support the existing business process: In the presence of dynamically changing attributes of the market that impact the demand (e.g., competitive prices), the retailer has to respond quickly to the changes in the market conditions. It is relatively easier, and often required, to more frequently change the online prices compared to brick prices, which typically incurs additional labor cost. Therefore, asynchronous channel-specific optimization becomes necessary along with the ability to execute rapid data refreshes. We demonstrate our ability to solve the integrated OCP on a weekly basis, and our algorithms were fast enough to support frequent re-optimization of online prices (one or more times a day or near real-time) while keeping the brick prices fixed at their most recent optimized values, thereby, accounting for their cross-channel impact. Analyzing and implementing a retailer’s competitive price response strategies in the omni-channel era is an important topic with research underway.

9.2. Post-deployment highlights

We conclude the paper some with important highlights. Our analytical solution was showcased as one of the retail analytics success stories in the smarter-commerce global summit in 2014. Today, several large global retail chains are regular users of the commercial offering including those with

whom we engaged. Overall, IBM has directly attributed several new market opportunities as well as significant incremental revenue to IBM in 2015 due to the deployed OCP solution. In November 2015, this work was formally recognized by IBM as one of the major accomplishments in 2015 by the research division.

Appendix A: Proof of Theorem 1

We drop subscript j in this proof as we are working with a specific location. We perform the market share variable transformation to obtain the solution of this problem in *near* closed form. Let θ_b denote the market share of brick as follows:

$$\theta_b = \frac{f_b(p_b)}{1 + f_b(p_b) + f_o(p_o)}. \quad (\text{A.1})$$

The lost market share and the online market share in terms of θ_b are then $\frac{1-\theta_b}{1+f_o(p_o)}$ and $\frac{f_o(p_o)(1-\theta_b)}{1+f_o(p_o)}$ respectively. Due to [assumption 1](#), the inverse attraction function defined as $g_b(y) = f_b^{-1}(y)$, $y > 0$ exists. Therefore, we can write $p_b = g_b\left(\frac{B\theta_b}{1-\theta_b}\right)$ where $B = 1 + f_o(p_o)$. Substituting this for p_b in problem OCP(p_o), we get

$$Z(\theta_b^*) = \max_{\theta_b \in [\Theta_L, \Theta_U]} A(1 - \theta_b) + \theta_b \left(g_b \left(\frac{B\theta_b}{1 - \theta_b} \right) - c_b \right) \quad (\text{A.2})$$

where $\Theta_U = \theta_b|_{p_b=\underline{h}(p_o)}$, $\Theta_L = \theta_b|_{p_b=\bar{h}(p_o)}$ and $A = (p_o - c_o)\frac{f_o(p_o)}{1+f_o(p_o)}$. Under [assumption 2](#), we know that this problem has a concave objective ([Keller et al. 2014](#)). Therefore, a solution to the equation that sets the first derivative of the objective to zero exists and it is an optimal solution to the unconstrained problem. We use this in deriving an optimal solution to the constrained problem as well. Now taking first derivative and setting it to zero, we get:

$$g_b \left(\frac{B\theta_b^*}{1 - \theta_b^*} \right) + \theta_b^* g_b' \left(\frac{B\theta_b^*}{1 - \theta_b^*} \right) B \frac{1}{(1 - \theta_b^*)^2} = A + c_b. \quad (\text{A.3})$$

Substituting $z = \frac{B\theta_b^*}{1 - \theta_b^*}$, we get

$$g_b(z) + z \left(1 + \frac{z}{B} \right) g_b'(z) = A + c_b$$

which is the same as [Eq. \(7.6\)](#).

Now substituting for $g_b(\cdot)$ in the objective function using the differential equation, the unconstrained maximum is

$$Z(\theta_b^*) = A - (A + c_b) \frac{z}{z + B} + \frac{z}{z + B} \left[A + c_b - \frac{z(z + B)}{B} g_b'(z) \right] = A - \frac{z^2}{B} g_b'(z) = A + H(p_o, \hat{p}_b).$$

The constrained optimal solution to the problem takes one of the following three values because the objective in the problem [\(A.2\)](#) is concave in θ_b : (a) optimal value, θ_b^* if $\Theta_L \leq \theta_b^* \leq \Theta_U$; (b)

Θ_U if $\theta_b^* > \Theta_U$; or (c) Θ_L if $\theta_b^* < \Theta_L$. In the price space, because θ_b and p_b have a one-to-one correspondence and have an inverse relationship, this optimal solution simplifies to Eq. (7.4) and translates to Eq. (7.5) for the optimal objective value.

Now for an MNL demand function, in particular, because $f_m(p_m) = e^{a_m - b_m p_m}$ for $m \in \{o, b\}$, $g_m(z) = \frac{a_m - \log z}{b_m}$ and $g'_m(z) = -\frac{1}{b_m z}$. Substituting this in Eq. (7.6) and setting $z = x(1 + f_o(p_o))$, we get the derivative and similarly the result for $H(p_o, \hat{p}_b)$. \square

Appendix B: Proof of Proposition 3

We drop subscript j in this proof as we are working with a specific location. Consider the derivative of $\psi(z, p_o)$ w.r.t. z :

$$\frac{\partial \psi(z, p_o)}{\partial z} = (2g'_b(z) + z g''_b(z)) \left(1 + \frac{z}{1 + f_o(p_o)} \right) \leq 0. \quad (\text{B.1})$$

The last inequality is because the first product term is always non-negative because of [assumption 2](#) and second product term is positive because z represents market share ratios and is always positive. This negative derivative implies that $\psi(z, p_o)$ is a non-increasing function in z .

From [assumption 1](#), it is easy to gather that the inverse function $g_b(z) = f_b^{-1}(p_b)$ will satisfy the following:

$$g'_b(z) \leq 0, \quad \lim_{z \rightarrow 0} g_b(z) = \infty, \quad \text{and} \quad \lim_{z \rightarrow \infty} g_b(z) = 0.$$

This implies, $\lim_{z \rightarrow 0} \psi(z, p_o) = \infty$, and $\lim_{z \rightarrow \infty} \psi(z, p_o) \leq 0$. In turn, this implies the differential Eq. (7.6) always has a solution because the right hand side is a positive constant for any given p_o .

Now consider the partial derivative of $\psi(z, p_o)$ w.r.t to p_o :

$$\frac{\partial \psi(z, p_o)}{\partial p_o} = -\frac{z^2 g'_b(z) f'_o(p_o)}{(1 + f_o(p_o))^2} \leq 0 \quad (\text{B.2})$$

The second inequality is because both g and f have negative derivatives because of [assumption 1](#). This negative derivative implies that $\psi(z, p_o)$ is a non-increasing function of p_o as well. \square

Appendix C: Proof of Proposition 1 and 2

We provide the proof of [Proposition 2](#) which considers the multi-location setting. The proof of [Proposition 1](#) is a special case of [Proposition 2](#) for a single location case and follows from this directly.

Consider a retailer in the absence any business rules. The OCP objective for the retailers is:

$$Obj = \sum_{j \in J} \tau_j \sum_{m \in M} (p_{mj} - c_{mj}) \frac{f_{mj}(p_{mj})}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} \Big|_{p_{mj} = p_m \quad \forall m \in V} \quad (\text{C.1})$$

We take the first derivative and set it equal to zero and obtain the following two conditions where $p_{mj} = p_m \forall m \in V$. :

$$\sum_{j \in J} \tau_j \left[\frac{f_{oj}(p_o) + (p_o - c_o) f'_{oj}(p_o)}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} - \sum_{m \in M} (p_{mj} - c_{mj}) \frac{f_{mj}(p_{mj}) f'_{oj}(p_o)}{[1 + \sum_{m' \in M} f_{m'j}(p_{m'j})]^2} \right] = 0 \quad \forall o \in V \quad (\text{C.2})$$

$$\frac{f_{bj}(p_{bj}) + (p_{bj} - c_{bj}) f'_{bj}(p_{bj})}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} - \sum_{m \in M} (p_{mj} - c_{mj}) \frac{f_{mj}(p_{mj}) f'_{bj}(p_{bj})}{[1 + \sum_{m' \in M} f_{m'j}(p_{m'j})]^2} = 0 \quad \forall b \in M \setminus V, j \in J \quad (\text{C.3})$$

Because the first order conditions are necessary conditions for optimality, all the optimal prices satisfy these conditions, maybe in addition to other prices.

Simplifying by substituting $\epsilon_{mj}(p_{mj}) = \frac{-f'_{mj}(p_{mj}) p_{mj}}{f_{mj}(p_{mj})}$ in the conditions we get

$$\sum_{j \in J} \tau_j \frac{f'_{oj}(p_o)}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} \left[\frac{p_o}{\epsilon_{oj}(p_o)} + (p_o - c_o) - \sum_{m \in M} (p_{mj} - c_{mj}) \frac{f_{mj}(p_{mj})}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} \right] = 0 \quad \forall o \in V \quad (\text{C.4})$$

$$\frac{p_{bj}}{\epsilon_{bj}(p_{bj})} + (p_{bj} - c_{bj}) - \sum_{m \in M} (p_{mj} - c_{mj}) \frac{f_{mj}(p_{mj})}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} = 0 \quad \forall b \in M \setminus V, j \in J \quad (\text{C.5})$$

Multiplying each of the latter location specific condition by $\tau_j \frac{f'_{oj}(p_o)}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})}$ and subtracting it from the first condition, we get,

$$\sum_{j \in J} \tau_j \frac{f'_{oj}(p_o)}{1 + \sum_{m' \in M} f_{m'j}(p_{m'j})} \left[(p_o - c_o) - (p_{bj} - c_{bj}) - \frac{p_o}{\epsilon_{oj}(p_o)} + \frac{p_{bj}}{\epsilon_{bj}(p_{bj})} \right] = 0 \quad (\text{C.6})$$

This proves [Proposition 2](#) in the multi-location setting.

Acknowledgments

First we thank the anonymous omni-channel retailers who engaged with us for this project. Next we thank all the people from IBM Commerce who guided us and helped commercialize OCP with a special thanks to Charles Tze Ng, Emrah Zarifoglu and Suzanne Valentine.

References

- Aggarwal, Gagan, Tomás Feder, Rajeev Motwani, An Zhu. 2004. Algorithms for multi-product pricing. *Automata, Languages and Programming*. Springer Berlin Heidelberg, 72–83.
- Akçay, Yalçın, Harihara Prasad Natarajan, Susan H Xu. 2010. Joint dynamic pricing of multiple perishable products under consumer choice. *Management Science* **56**(8) 1345–1361.
- Ansari, Asim, Carl F Mela, Scott A Neslin. 2008. Customer channel migration. *Journal of Marketing Research* **45**(1) 60–76.

- Avery, Jill, Thomas J Steenburgh, John Deighton, Mary Caravella. 2012. Adding bricks to clicks: Predicting the patterns of cross-channel elasticities over time. *Journal of Marketing* **76**(3) 96–111.
- Aydin, Goker, Evan L Porteus. 2008. Joint inventory and pricing decisions for an assortment. *Operations Research* **56**(5) 1247–1255.
- Bell, D. R., S. Gallino, A. Moreno. 2014. How to win in an omnichannel world. *MIT Sloan Management Review* .
- Ben-Akiva, Moshe E, Steven R Lerman. 1985. *Discrete choice analysis: theory and application to travel demand*, vol. 9. MIT press.
- Berry, Steven T. 1994. Estimating discrete-choice models of product differentiation. *The RAND Journal of Economics* 242–262.
- Bitran, Gabriel, René Caldentey. 2003. An overview of pricing models for revenue management. *Manufacturing & Service Operations Management* **5**(3) 203–229.
- Brynjolfsson, E., Y. J. Hu, M. S. Rahman. 2013. Competing in the age of omnichannel retailing. *MIT Sloan Management Review* .
- Business Insider. 2015. The us digital media ad spend report: Mobile will fuel strong growth across formats as desktop and traditional spend slow. URL <http://www.businessinsider.com/how-ad-spending-on-mobile-will-grow-faster-than-on-any-other-advertising-channel-2015-7>.
- Charnes, Abraham, William W Cooper. 1962. Programming with linear fractional functionals. *Naval Research logistics quarterly* **9**(3-4) 181–186.
- Chen, Xin, David Simchi-Levi. 2012. *The Oxford handbook of pricing management*, chap. Pricing and inventory management. Oxford University Press, 784–822.
- Chintagunta, Pradeep K, Junhong Chu, Javier Cebollada. 2012. Quantifying transaction costs in online/off-line grocery channel choice. *Marketing Science* **31**(1) 96–114.
- Dong, Lingxiu, Panos Kouvelis, Zhongjun Tian. 2009. Dynamic pricing and inventory control of substitute products. *Manufacturing & Service Operations Management* **11**(2) 317–339.
- Elmaghraby, Wedad, Pinar Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science* **49**(10) 1287–1309.
- Friedman, Jerome, Trevor Hastie, Rob Tibshirani. 2010. Regularization paths for generalized linear models via coordinate descent. *Journal of statistical software* **33**(1) 1.
- Gallego, Guillermo, Ruxian Wang. 2014. Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Operations Research* **62**(2) 450–461.
- Gallino, Santiago, Antonio Moreno. 2014. Integration of online and offline channels in retail: The impact of sharing reliable inventory availability information. *Management Science* **60**(6) 1434–1451.

- Hanson, Ward, Kipp Martin. 1996. Optimizing multinomial logit profit functions. *Management Science* **42**(7) 992–1003.
- Hua, Guowei, Shouyang Wang, TC Edwin Cheng. 2010. Price and lead time decisions in dual-channel supply chains. *European journal of operational research* **205**(1) 113–126.
- Huffington Post. 2013. Top retail buzzwords for 2013: Omnichannel, personalization, mobile. URL http://www.huffingtonpost.com/andrew-cherwenka/top-retail-buzzwords-for-_b_2506997.html.
- Internet Retailer. 2015. E-commerce sales grow six times faster for u.s. top 500 merchants than total retail sales. URL <https://www.internetretailer.com/2015/04/13/e-commerces-sales-outgrow-total-retail-sales-2014>.
- Keller, Philipp W, Retsef Levi, Georgia Perakis. 2014. Efficient formulations for pricing under attraction demand models. *Mathematical Programming* **145**(1-2) 223–261.
- Kireyev, Pavel, Vineet Kumar, Elie Ofek. 2014. Match your own price? self-matching as a retailers multi-channel pricing strategy. Tech. rep., Working Paper, Harvard Business School.
- Li, Hongmin, Woonghee Tim Huh. 2011. Pricing multiple products with the multinomial logit and nested logit models: Concavity and implications. *Manufacturing & Service Operations Management* **13**(4) 549–563.
- MacQueen, James, et al. 1967. Some methods for classification and analysis of multivariate observations. *Proceedings of the fifth Berkeley symposium on mathematical statistics and probability*, vol. 1. 281–297.
- McFadden, D. 1974. Conditional logit analysis of qualitative choice behavior. *Frontiers in Econometrics* 105–142.
- Newman, Jeffrey P., Mark E. Ferguson, Laurie A. Garrow, Timothy L. Jacobs. 2014. Estimation of choice-based models using sales data from a single firm. *Manufacturing and Service Operations Management* **16**(2) 184–197.
- Pisinger, David. 1994. A minimal algorithm for the multiple-choice knapsack problem. *European Journal of Operational Research* **83** 394–410.
- Rayfield, W Zachary, Paat Rusmevichientong, Huseyin Topaloglu. 2012. Approximation methods for pricing problems under the nested logit model with price bounds. Working paper.
- Reibstein, David J, Hubert Gatignon. 1984. Optimal product line pricing: The influence of elasticities and cross-elasticities. *Journal of Marketing Research* **21**(3).
- Retail Dive. 2015. A chilly black friday for some brick-and-mortar retailers online sales appear strong as line blurs between shopping experiences. URL <http://www.retaildive.com/news/a-chilly-black-friday-for-some-brick-and-mortar-retailers/409884/>.
- Rusmevichientong, Paat, David Shmoys, Huseyin Topaloglu. 2010. Assortment optimization with mixtures of logits. Tech. rep., Tech. rep., School of IEOR, Cornell University.

- Rusmevichientong, Paat, Benjamin Van Roy, Peter W Glynn. 2006. A nonparametric approach to multi-product pricing. *Operations Research* **54**(1) 82–98.
- Schön, Cornelia. 2010. On the product line selection problem under attraction choice models of consumer behavior. *European Journal of Operational Research* **206**(1) 260–264.
- Sherali, Hanif D, Warren P Adams. 1999. Reformulation-linearization techniques for discrete optimization problems. *Handbook of combinatorial optimization*. Springer, 479–532.
- Shopify. 2015. E-commerce platform vendor shopify grows q3 revenue 93%. Internet Retailer. URL <https://www.internetretailer.com/2015/11/09/e-commerce-platform-vendor-shopify-grows-q3-revenue-93>.
- Song, Jing-Sheng, Zhengliang Xue. 2007. Demand management and inventory control for substitutable products. *Working paper* .
- Subramanian, Shivaram, Pavithra Harsha. 2015. An integrated estimation method for predicting demand and lost sales in consumer choice models. Working paper, IBM Research.
- Subramanian, Shivaram, Hanif D. Sherali. 2010. A fractional programming approach for retail category price optimization. *Journal of Global Optimization* **48**(2) 263–277.
- Talluri, Kalyan. 2009. A finite-population revenue management model and a risk-ratio procedure for the joint estimation of population size and parameters. Tech. rep., Universitat Pompeu Fabra.
- Talluri, Kalyan, Garrett Van Ryzin. 2004. Revenue management under a general discrete choice model of consumer behavior. *Management Science* **50**(1) 15–33.
- Train, Kenneth E. 2009. *Discrete choice methods with simulation*. 2nd ed. Cambridge university press.
- Urban, Glen L. 1969. A mathematical modeling approach to product line decisions. *Journal of Marketing Research* 40–47.
- WAL. 2015. Walmart report. URL <http://news.walmart.com/news-archive/2015/06/05/walmart-ceo-to-associates-you-are-our-heroes>. Share holders meeting.
- Yan, Ruiliang, Zhi Pei. 2011. Information asymmetry, pricing strategy and firm’s performance in the retailer-multi-channel manufacturer supply chain. *Journal of Business Research* **64**(4) 377–384.