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## Tiling 2-Deficient Rectangular Solids with L-Trominoes in Three and Higher Dimensions

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# Tiling 2-Deficient Rectangular Solids with L-Trominoes in Three and Higher Dimensions

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## Abstract

In 1954 Solomon Golomb [4] showed that if you remove a square from a chess board of size $2^N x 2^N$ then the resulting board can always be tiled by L-shaped trominoes (polyominoes of three squares). In the 1980s Chu and Johnsonbaugh [2, 3] characterized which 2D $MxN$ boards are tilable by L-tromines if you remove an arbitrary square. In recent work [1], the current authors showed that arbitrary 3D rectangular boards with one cube removed, of dimension $KxLxM$, where $KLM \equiv 1 \pmod 3$ and $K,L,M > 1$ are L-tilable. We also extended this result to all higher dimensions.

2-deficient boards (boards with two squares or cubes removed) are especially interesting because it is easy to see that there are **no** rectangular $NxM$ boards that are generically tilable in the sense that they can be tiled with L-trominoes regardless of the squared removed – just remove two squares that effectively isolate a corner square. However, in 2008 Starr [7] showed that all 3D cubical boards of dimension $NxNxN$ for $N \equiv 2 \pmod 3$ with two cubes removed are L-tilable. In the present work we extend Starr's result to show that indeed the same is true for arbitrary 3D rectangular boards of dimension $KxLxM$, where $KLM \equiv 2 \pmod 3$, and $K,L,M > 1$. As in our earlier work, we extend the result to all higher dimensions.


## 1 L-Trominoes in 3D

A polyomino is a finite collection of edge-connected, equal-sized, squares in the plane. 3D or solid polyominoes are face-connected, equal-sized cubes. A tromino is a polyomino consisting of just three squares, or in 3D, a polyomino consisting of three cubes. We are particularly interested in so-called L-trominoes. See Figure 1.

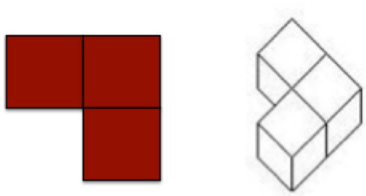

**Figure 1.** 2D and 3D L-trominoes.

A classic theorem of Solomon Golomb from 1954 [4] says that if you remove a square from a chess board of size $2^N x 2^N$ then the resulting board can always be tiled by L-trominoes. Since Golomb's 1954 paper the tiling of arbitrary sized rectangular planar boards with one or two squares removed has been extensively studied. Such boards are called 1-deficient and 2-deficient, respectively. In 2008, Starr [7] showed that any 1-deficient cubical board with edge length $N \equiv 1 \pmod 3$, or 2-deficient cubical board with edge length $N \equiv 2 \pmod 3$ could be tiled by solid L-trominoes. Recently the current authors [1] extended the 1-deficient result on cubical boards to show that an arbitrary board of size $KxLxM$, where $KLM \equiv 1 \pmod 3$ and $K,L,M > 1$ are L-tilable. The purpose of this notes is to sketch the same result for the case where $KLM \equiv 2 \pmod 3$ and $K,L,M > 1$.

We shall require the following result of Boltyanski and Soifer (Theorem 20.1 in [6]):

**Lemma 1** *Any board of size $KxLxM$ where $3 \mid KLM$ and $K,L,M \geq 2$ is L-tilable.*

And we restate our main result from [1], which we shall frequently make use of:

**Theorem 2** *A $(KxLxM)$-1 board, where $KLM \equiv 1 \pmod 3$ and $K,L,M > 1$ is always L-tilable.*

If we denote our board in general terms as a $(K\text{x}L\text{x}M)$-2 board, where $KLM \equiv 2 \pmod 3$ and $K,L,M > 1$ then the argument will be a triple induction on $K$, $L$ and $M$. In this argument there are four essentially "prime" cases that need to be argued independently to kick off the induction, these being the cases where all values of $K,L,M \in \{2,5\}$, so the cases $(2x2x2)-2$, $(2x2x5)-2$, $(2x5x5)-2$, and $(5,5,5)-2$. The cases $(2x2x2)-2$ and $(5,5,5)-2$ are handled by Starr's Theorem [7] on L-tiling of generic $N^3-1$ boards. We omit the other two cases for the sake of brevity. In short order it will become evident why these are the "prime" cases.

First let us suppose that there is just one of the dimensions from among the $K,L,M$ that is not in $\{2,5\}$, and without loss of generality suppose it is $K$. Since $K \equiv 2 \pmod 3$, we must have $K \geq 8$. Now either some $(K-3)xLxM$ sub-board contains both missing cells and we break the board up into a disjoint union of this $(K-3)xLxM$ board and a $3xLxM$ board - for short we shall refer to this split using the shorthand $KxLxM = (K-3)xLxM \sqcup 3xLxM$, or we can break the board up into the disjoint union $(K-4)xLxM \sqcup 4xLxM$ where each sub-board contains one missing cell. In the former case we use induction to L-tile the d-deficient $(K-3)xLxM$ board and Lemma 1 to L-tile the $3xLxM$ board. In the latter case we can tile each of the two 1-deficient boards using Theorem 2.


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Next suppose that some (at least) two of the $K,L,M \notin \{2,5\}$ and argue by double induction[1]. Say the two are $L,M$, i.e. $L,M \in \{4,7,8,10,...\}$.

If now $L \equiv 1$ (mod 3), write

$L = L_0 + L_1$ where $L_0, L_1 \equiv 2$ (mod 3), i.e. $L_0 = L-2, L_1 = 2$,

while if $L \equiv 2$ (mod 3), write

$L = L_0 + L_1$ where $L_0, L_1 \equiv 1$ (mod 3), i.e. $L_0 = L-4, L_1 = 4$.

Similarly write $M = M_0 + M_1$ using the same convention.

Then we have two ways of splitting the solid, either

$$\begin{aligned} KxLxM &= KxL_0xM \sqcup KxL_1xM, \text{ or} \qquad (1)\\ &= KxLxM_0 \sqcup KxLxM_1 \qquad (2)\end{aligned}$$

where $KL_0M, KL_1M, KLM_0, KLM_1 \equiv 1$ (mode 3). Moreover, we have the decomposition

$$KxLxM = KxL_0xM_0 \sqcup KxL_0xM_1 \sqcup KxL_1xM_0 \sqcup KL_1xM_1. \quad (3)$$

We illustrate these decompositions in Figure 2 for the case $KxLxM = 2x4x7$, which we visualize as two parallel copies of a $4x7$ board. If we make both the cuts illustrated in this figure we get a decomposition $2x4x7 = 2x2x2 \sqcup 2x2x2 \sqcup 2x2x5 \sqcup 2x2x5$, which corresponds to the more general Equation 3.

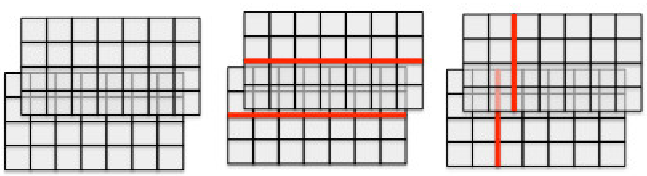

**Figure 2.** A $2x4x7$ board visualized as two parallel $2x4$ boards (left). The same board visualized with a cut corresponding to the decomposition $2x4x7 = 2x2x7 \sqcup 2x2x7$ (center) and $2x4x7 = 2x4x2 \sqcup 2x4x5$ (right).

Now note that if we remove two cells from the $KxLxM$ board that either they belong one to each side of the split associated with Equations 1 or 2, OR both cells belong to one of the smaller sub-boards corresponding to Equation 3. In the first case we can tile the two sub-boards, and hence the entire $(KxLxM)-2$ board using Theorem 2. In the latter case the small sub-board can be L-tiled by induction and it remains to L-tile the remaining three sub-boards. We denote these sub-boards, for convenience by $A,B,C$ with $A$ connected to $B$ and $B$ connected to $C$.

We L-tile $A \sqcup B \sqcup C$ as follows. Place an L-tromino so that it takes two cells from the face of $A$ that is adjacent to $B$ and one cell from the face of $B$ that is adjacent to $A$. Analogously take two cells out of the face of $C$ that is adjacent to $B$ and once cell from the face of $B$ that is adjacent to $C$, taking care not to take the same cell as earlier (there are at least 4 cells in these faces so this cannot be a problem). If we take away the cells with these trominoes, $A,B$ and $C$ are each left with two cells removed and then can be further L-tiled by induction, completing the tiling and the argument. We thus have:

**Theorem 3** *A $(KxLxM)$-2 board, where $KLM \equiv 2$ $(mod\ 3)$ and $K,L,M > 1$ is always L-tilable.*

[1] Implicitly this argument is actually a triple induction.

## 2 L-Trominoes in Higher Dimensions

We can use the same trick we used in [1] to embed a 2-deficient board in 4D as a 2-deficient board in 3D such that all adjacencies of cells in the 3D board are adjacencies in the 4D board (though obviously not vice versa). Since this embedding can be extended inductively to higher and higher dimensions we obtain the following generalization of Theorem 3.

**Theorem 4** *A $(K_1x\cdots xK_N)$-2 board, where $K_1\cdots K_N \equiv 2(mod\ 3)$, for $N \geq 3$ and some three of the $K_i > 1$, is always L-tilable.*

## 3 Future Work

It is not possible to L-tile an arbitrary 3-deficient board in 3D since one can remove three cells and isolate the corner cell. However, we conjecture that it is always possible to L-tile a 3-deficient board in 4D and higher for board-sizes of the needed modularity and wonder if this isolation of the corner cell is the only way to obstruct an L-tiling in 3D. In general, for a given number $K$ can one find a minimum dimension $N$ such that *any* $K$-deficient board in $N$ dimensions is L-tilable?

## References

[1] Arthur Befumo and Jonathan Lenchner. Tiling one-deficient rectangular solids with trominoes in three and higher dimensions. *Mathematics Magazine*, To Appear.

[2] I-P. Chu and R. Johnsonbaugh. Tiling boards with trominoes. *Journal of Recreational Mathematics*, 18:188–193, 1985-86.

[3] I-P. Chu and R. Johnsonbaugh. Tiling deficient boards with trominoes. *Mathematics Magazine*, 59:34–40, 1986.

[4] Solomon W. Golomb. Checker boards and polyominoes. *Amer. Math. Monthly*, 61:675–682, 1954.

[5] George E. Martin. *Polyominoes. A Guide to Puzzles and Problems in Tiling*. Mathematical Association of America, 1991.

[6] Alexander Soifer. *Geometric Etudes in Combinatorial Mathematics*. Springer, 2nd edition, 2010.

[7] Norton Starr. Tromino tiling deficient cubes of side length $2^n$. *Geombinatorics*, XVIII(2):72–87, 2008.