RC25634 (WAT1610-092) October 28, 2016
Mathematics

IBM Research Report

Lattice Closures of Polyhedra

Sanjeeb Dash, Oktay Günlük
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598 USA

Diego A. Morán
Universidad Adolfo Ibañez

Lattice closures of polyhedra

Sanjeeb Dash
IBM Research
sanjeebd@us.ibm.com

Oktay Günlük
IBM Research
gunluk@us.ibm.com

Diego A. Morán R.
Universidad Adolfo Ibañez
diego.moran@uai.cl

October 27, 2016

Abstract

We define the k-dimensional lattice closure of a polyhedral mixed-integer set to be the intersection of the convex hulls of all possible relaxations of the set obtained by choosing up to k integer vectors $\pi_1, \ldots, \pi_k$ and requiring $\langle \pi_1, x\rangle, \ldots, \langle \pi_k, x\rangle$ to be integral. We show that given any collection of such relaxations, finitely many of them dominate the rest. The k-dimensional lattice closure is equal to the split closure when $k = 1$. Therefore the k-dimensional lattice closure of a rational polyhedral mixed-integer set is a polyhedron when $k = 1$ and our domination result extends this to all $k \geq 2$. We also construct a polyhedral mixed-integer set with $n > k$ integer variables such that finitely many iterations of the k-dimensional lattice closure do not give the integer hull. In addition, we use this result to show that t-branch split cuts cannot give the integer hull, nor can valid inequalities from unbounded, full-dimensional, convex lattice-free sets.

1 Introduction

Cutting planes (or *cuts*, for short) are linear inequalities satisfied by the integral points in a polyhedron. In practice, cutting planes are used to give a tighter approximation of the convex hull of integral solutions of a mixed-integer program (MIP) than the LP relaxation. A widely studied family of cutting planes is the family of *Split cuts*, and special classes of split cuts, namely *Gomory mixed-integer cuts* and *Zero-half Gomory-Chvátal cuts*, are very effective in practice and are used by commercial MIP solvers.

A split cut for a polyhedron $P \subseteq \mathbb{R}^n$ is a linear inequality $c^T x \leq d$ that is valid for

$$P \setminus \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}$$

for some $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$ (we call $\{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}$ a *split set*). If P is the continuous relaxation of a mixed-integer set and π has non-zero coefficients only for the indices that correspond to integer variables, then the resulting inequality is valid for the mixed-integer set.

An important theoretical question for a family of cuts for a polyhedron is whether only finitely many cuts from the family imply the rest. Cook, Kannan and Schrijver [11] proved that the split closure of a rational polyhedron – the set of points that satisfy all split cuts – is again a polyhedron, thus showing that only finitely many split cuts for a rational polyhedron imply the remaining split cuts. Furthermore, they also give a polyhedral mixed-integer set with unbounded *split rank* – the convex hull of points cannot be obtained by finitely repeating the split closure operation starting from the natural polyhedral relaxation of the mixed-integer set. Earlier, Schrijver [26] showed that the set of points in a rational polyhedron satisfying all Gomory-Chvátal cuts is a polyhedron, and Dunkel and Schulz [21] and Dadush, Dey and Vielma [13] proved that this result holds, respectively, for arbitrary polytopes, and compact convex sets.

Recently there has been a significant amount of research on generalizing split cuts in different ways to obtain new and more effective classes of cutting planes. Andersen, Louveaux, Weismantel and Wolsey [3] studied *lattice-free cuts* in the context of the two-row continuous group relaxation and demonstrated that these cuts generalize split cuts. They obtain lattice-free cuts from two dimensional convex lattice-free sets, and observe that split cuts are obtained from a family of two-dimensional lattice-free polyhedra with two parallel sides. Basu, Hildebrand and Köeppe [8] showed that the triangle closure (points satisfying cuts obtained from maximal lattice-free triangles in $\mathbb{R}^2$) of the two-row continuous group relaxation is a polyhedron, and we showed in [18] that the quadrilateral closure is also a polyhedron. Furthermore, Andersen, Louveaux and Weismantel [2] showed that the set of points in a rational polyhedron satisfying all cuts obtained from convex, lattice-free sets with bounded max-facet-width is a polyhedron.

As a different generalization of split cuts, Li and Richard [24] defined *t-branch split cuts* which are obtained by considering t split sets simultaneously, where t is a positive integer. In particular, a t-branch split cut for a polyhedron P is a linear inequality valid for $P \setminus \cup_{i=1}^{t} S_i$ where S_i is a split set for $i = 1, \ldots, t$. The 1-branch split cuts are equivalent to the family of split cuts studied by Cook, Kannan, and Schrijver. Li and Richard also constructed a polyhedral mixed-integer set that has unbounded 2-branch split rank, i.e., repeating the 2-branch split closure operation does not yield the convex hull of the points in the mixed-integer set. Polyhedral mixed-integer sets with unbounded t-branch split rank for any fixed $t > 2$ were given in [16]. We proved in [18] that the t-branch split closure of a rational polyhedron is a polyhedron for $t = 2$. We later extended this result to any integer $t > 0$ in [19]. Furthermore, we also studied cuts obtained by simultaneously considering t convex lattice-free sets with bounded max-facet-width, and showed that the associated closure is a polyhedron.

In this paper, we study an alternative method of generalizing split cuts and prove that the associated closures are also polyhedral, if one starts from a rational polyhedron. Cook, Kannan, and Schrijver [11] gave an alternative definition (to the one given earlier) of split cuts: they define a split cut for $P \subseteq \mathbb{R}^n$ to be a linear inequality valid for

$$\{x \in P : \pi^T x \in \mathbb{Z}\} \ = \ \bigcup_{\pi_0=-\infty}^{\infty} \{x \in P : \pi^T x = \pi_0\}$$

for some $\pi \in \mathbb{Z}^n$. We generalize this idea by considering valid linear inequalities for sets of the form

$$\{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_k^T x \in \mathbb{Z}\}, \tag{1}$$

for some $\{\pi_1, \ldots, \pi_k\} \subseteq \mathbb{Z}^n$ where k is a fixed positive integer. We call these cutting planes *k-dimensional lattice cuts* (we will explain the motivation for this name shortly). Clearly, when

$k = 1$, the resulting cuts are split cuts, according to the definition of Cook, Kannan and Schrijver.

In this paper, we prove that for a rational polyhedron and a fixed integer k, the *k-dimensional lattice closure* of P – the set of points satisfying all k-dimensional lattice cuts – is a polyhedron. In fact, we prove the following more general result: Given a rational polyhedron P, a fixed positive integer k, and an arbitrary collection $\mathcal{L}$ of tuples of the form $(\pi_1, \ldots, \pi_k)$ with $\pi_i \in \mathbb{Z}^n$, we show that there exists a finite $\mathcal{F} \subseteq \mathcal{L}$ with the property that for any $(\pi_1, \ldots, \pi_k) \in \mathcal{L}$, there is a tuple $(\mu_1, \ldots, \mu_k) \in \mathcal{F}$ such that

$$\text{conv}(\{x \in P : \mu_1^T x \in \mathbb{Z}, \ldots, \mu_k^T x \in \mathbb{Z}\}) \subseteq \text{conv}(\{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_k^T x \in \mathbb{Z}\}).$$

In other words, the k-dimensional cuts obtained from the tuple $(\mu_1, \ldots, \mu_k)$ imply all such cuts obtained from $(\pi_1, \ldots, \pi_k)$. Together with the fact that

$$\text{conv}(\{x \in P : \mu_1^T x \in \mathbb{Z}, \ldots, \mu_k^T x \in \mathbb{Z}\})$$

is a polyhedron for any integral $\mu_1, \ldots, \mu_k$, the polyhedrality result above follows.

Dash, Dey and Günlük [14] defined a generalization of 2-branch split cuts called *crooked cross cuts*. Furthermore, Dash, Dey and Günlük [15] studied 2-dimensional lattice cuts and showed that they were equivalent to the family of crooked cross cuts. The results in this paper show that the crooked cross closure of a rational polyhedron is also a polyhedron.

We also construct a polyhedral set that has unbounded rank with respect to the k-dimensional lattice closure. This latter result implies that the same polyhedral set has unbounded rank with respect to k-branch split cuts, which was earlier proved in [16]. More generally, this implies that this polyhedral set has unbounded rank with respect to cuts obtained from all unbounded, full-dimensional, maximal, convex lattice-free sets.

In the next section, we formally define split cuts and k-dimensional lattice cuts in the context of polyhedral mixed-integer sets. In Section 3, we use the notion of well-ordered qosets to define a dominance relationship between lattice cuts. In Section 4, we define lattice closures, and show that the lattice closure of a rational polytope is a polytope, and we extend this result to unbounded polyhedra in Section 5. In Section 6, we show that for any $n > 1$, there is a polyhedral mixed-integer set with n integer variables and one continuous variable such that the integer hull cannot be obtained by finitely iterating the k-dimensional lattice closure for $k < n$.

2 Preliminaries

For a given set $X \subseteq \mathbb{R}^n$, we denote its convex hull by $\text{conv}(X)$. Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron (all polyhedra in this paper are assumed to be rational). Let $0 \le l \le n$ and $I = \{1, \ldots, l\}$. In what follows, we will think of I as the index set of variables restricted to be integral. A set of the form

$$P^I = \{x \in P : x_i \in \mathbb{Z}, \text{ for } i \in I\}$$

is a polyhedral mixed-integer set, and we call P the linear relaxation of P^I. Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, where the last $n - l$ components of π are zero, the *split set* associated with (π, π_0) is defined to be

$$S(\pi, \pi_0) = \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}.$$

We refer to a valid inequality for $\text{conv}(P \setminus S(\pi, \pi_0))$ to be a split cut for P derived from $S(\pi, \pi_0)$. As $\pi \in \mathbb{Z}^l \times \{0\}^{n-l}$, it follows that

$$\mathbb{Z}^l \times \mathbb{R}^{n-l} \subseteq \mathbb{R}^n \setminus S(\pi, \pi_0),$$

and therefore split cuts derived from the associated split sets are valid for the mixed-integer set P^I.

Let $\mathcal{S}^1 = \{S(\pi, \pi_0) : \pi \in \mathbb{Z}^l \times \{0\}^{n-l}, \pi_0 \in \mathbb{Z}\}$; in other words $\mathcal{S}^1$ is the set of all possible split sets in $\mathbb{R}^n$ that lead to valid inequalities for P^I. Let $\mathcal{S} \subseteq \mathcal{S}^1$. We define the *split closure of P with respect to $\mathcal{S}$* as

$$\text{SC}(P, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \text{conv}\,(P \setminus S)\,.$$

We call $\text{SC}(P, \mathcal{S}^1)$ *the split closure of P*. Cook, Kannan and Schrijver [11] proved that $\text{SC}(P, \mathcal{S}^1) = \text{SC}(P, \mathcal{F})$ for some finite set $\mathcal{F} \subset \mathcal{S}^1$. Later Andersen, Cornuéjols and Li [1] extended this result by showing that the same result holds if one replaces $\mathcal{S}^1$ with an arbitrary set $\mathcal{S} \subseteq \mathcal{S}^1$.

Given a positive integer t, we define a t-branch split set in $\mathbb{R}^n$ to be a set of the form $\cup_{i=1}^t S_i$, where $S_i \in \mathcal{S}^1$. Note that we allow repetition of split sets in this definition. Let $\mathcal{S}^t$ denote the set of all possible t-branch split sets in $\mathbb{R}^n$, and let $\mathcal{T} \subseteq \mathcal{S}^t$. We define

$$\text{Cl}(P, \mathcal{T}) = \bigcap_{T \in \mathcal{T}} \text{conv}\,(P \setminus T)\,,$$

and call $\text{Cl}(P, \mathcal{T})$ the *t-branch split closure of P* with respect to $\mathcal{T}$. We proved in [19] that for any $\mathcal{T} \subseteq \mathcal{S}^t$ there exists a finite subset $\mathcal{F}$ of $\mathcal{T}$ such that for any $T \in \mathcal{T}$, there is a $T' \in \mathcal{F}$ satisfying $\text{conv}(P \setminus T') \subseteq \text{conv}(P \setminus T)$. In other words, given any family $\mathcal{T}$ of t-branch split sets, there is a finite subfamily where cuts obtained from an element of $\mathcal{T}$ are dominated by cuts from an element of the finite sublist. This result generalizes Averkov's result [4] on split sets. Further, our result above implies that the $\text{Cl}(P, \mathcal{T})$ is a polyhedron for any $\mathcal{T} \subseteq \mathcal{S}^t$, thus generalizing the split closure result of Cook, Kannan and Schrijver.

Cook, Kannan and Schrijver [11] gave an alternative definition of the split closure which is equivalent to the one above:

$$\text{SC}(P, \mathcal{S}^1) = \bigcap_{\pi \in \mathbb{Z}^l \times \{0\}^{n-l}} \text{conv}\left(\{x \in P : \pi^T x \in \mathbb{Z}\}\right) \tag{2}$$

As discussed in the introduction, a natural way of generalizing this definition of the split closure is as follows. Let Π^k be the collection of all tuples of the form $(\pi_1, \ldots, \pi_k)$ where $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$ for $i = 1, \ldots, k$. As $x \in \mathbb{Z}^l \times \mathbb{R}^{n-l}$ implies that $\pi_i^T x$ is integral, it follows that for any $\tilde{\Pi} \subseteq \Pi^k$, P^I is contained in the set

$$\text{Cl}(P, \tilde{\Pi}) = \bigcap_{(\pi_1, \ldots, \pi_k) \in \tilde{\Pi}} \text{conv}\left(\{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_k^T x \in \mathbb{Z}\}\right).$$

Now consider $k = 2$ and let $\pi_1, \pi_2 \in \mathbb{Z}^n$ and q be a nonzero integer. It is easy to see that

$$\{x : \pi_1^T x \in \mathbb{Z}, \pi_2^T x \in \mathbb{Z}\} = \{x : \pi_1^T x \in \mathbb{Z}, (\pi_2 + q\pi_1)^T x \in \mathbb{Z}\}.$$

In other words, (π_1, π_2) does not uniquely define the set

$$\{x \in \mathbb{R}^n : \pi_1^T x \in \mathbb{Z}, \pi_2^T x \in \mathbb{Z}\}. \tag{3}$$

Furthermore, the set in (3) is a *mixed-lattice*, and we will next delve into basic lattice theory in order to understand representability issues for a set of the form (3).

2.1 Lattices

For a linear subspace V of $\mathbb{R}^n$, $V^\perp$ denotes the orthogonal complement of V, i.e., $V^\perp = \{x \in \mathbb{R}^n \,:\, x^T y = 0 \text{ for all } y \in V\}$. The projection of a set $S \subseteq \mathbb{R}^n$ onto V is $\text{Proj}_V(S) = \{x \in V \,:\, \exists y \in V^\perp \text{ such that } x + y \in S\}$. Let $\{c_1, \ldots, c_m\}$ be a set of rational vectors in $\mathbb{R}^n$. The span of $\{c_1, \ldots, c_m\}$ is the linear subspace of $\mathbb{R}^n$ consisting of all linear combinations of the set of vectors:

$$\text{span}(c_1, \ldots, c_m) = \{x \in \mathbb{R}^n \,:\, x = a_1 c_1 + \cdots + a_m c_m,\ a_i \in \mathbb{R}\}.$$

The lattice generated by $\{c_1, \ldots, c_m\}$ is the set of all integer linear combinations of these vectors:

$$\text{Lat}(c_1, \ldots, c_m) = \{x \in \mathbb{R}^n \,:\, x = u_1 c_1 + \cdots + u_m c_m,\ u_i \in \mathbb{Z}\}.$$

Throughout this paper, we will be interested only in *rational lattices* and *rational linear subspaces*, i.e., lattices and subspaces that are generated by rational vectors.

The dimension of the lattice $L = \text{Lat}(c_1, \ldots, c_m)$, denoted by $\dim(L)$, is equal to the dimension of the linear subspace spanned by the vectors in L and there always exists exactly $\dim(L)$ linearly independent vectors that generate the lattice L. Any set of linearly independent vectors in L that generate L is called a basis. Every basis of a lattice has the same cardinality, and any lattice with dimension two or more has infinitely many bases. If $\{b_1, \ldots, b_k\}$ is a basis of L, the matrix whose columns are $b_1, \ldots, b_k$ is commonly called a *basis matrix* of L.

If $L \subseteq \mathbb{R}^n$ is a lattice, then its *dual lattice* is denoted by L^* and is defined as

$$L^* = \{x \in \text{span}(L) : y^T x \in \mathbb{Z} \text{ for all } y \in L\},$$

and it has the property that

$$(L^*)^* = L.$$

In the definition of L^* above, it suffices to only consider a set of $y \in L$ that generate L; i.e.,

$$\text{Lat}(b_1, \ldots, b_k)^* = \{x \in \text{span}(b_1, \ldots, b_k) \,:\, b_i^T x \in \mathbb{Z} \text{ for } i = 1, \ldots, k\}.$$

If B is a basis matrix of L, then $B(B^T B)^{-1}$ is a basis matrix of L^*.

We define a *mixed lattice* in $\mathbb{R}^n$ as a set of the form $L + \text{span}(L)^\perp$ where L is a lattice in $\mathbb{R}^n$. For a mixed lattice $M = L + \text{span}(L)^\perp$, we say that L is the underlying lattice and M has *lattice-dimension* $\dim(L)$.

For $\pi \in \mathbb{Z}^n \setminus \{0\}$, let

$$M(\pi) = \{x \in \mathbb{R}^n \,:\, \pi^T x \in \mathbb{Z}\}.$$

Note that $M(\pi)$ is a rational mixed-lattice, as

$$M(\pi) = \{x \in \mathbb{R}^n \;:\; x = q\,\frac{\pi}{||\pi||^2} + v,\; q \in \mathbb{Z},\; v \in V\}$$

where $V = \text{span}(\pi)^\perp$ and $\|\cdot\|$ denotes the usual Euclidian norm. We say that $M(\pi)$ is a mixed-lattice in $\mathbb{R}^n$ defined by π and its lattice-dimension is 1. We define

$$\mathcal{M}_n^1 = \{M(\pi) \;:\; \pi \in \mathbb{Z}^n \setminus \{0\}\}$$

and

$$\mathcal{M}_n^k = \big\{ \cap_{j=1}^k M_j \;:\; M_j \in \mathcal{M}_n^1 \text{ for all } j \in \{1, \ldots, k\}\big\}.$$

Clearly all $M(\pi)$ contain $\mathbb{Z}^n$ and therefore any $M \in \mathcal{M}_n^k$ contains $\mathbb{Z}^n$. Conversely, any mixed lattice $M \subset \mathbb{R}^n$ of lattice dimension k that contains $\mathbb{Z}^n$ is an element of $\mathcal{M}_n^k$. Throughout the paper we will use $\mathcal{M}^k$ instead of $\mathcal{M}_n^k$ when n is clear from the context.

Note that the expression in (2) can be written as

$$\bigcap_{\pi \in \mathbb{Z}^n \setminus \{0\}} \text{conv}(P \cap M(\pi)).$$

Furthermore, the set in (3) can be written as $M(\pi_1) \cap M(\pi_2)$ and is a mixed-lattice. More generally, any $M = \cap_{i=1}^k M(\pi_i) \in \mathcal{M}^k$ can be written as

$$M = L + \text{span}(\pi_1, \ldots, \pi_k)^\perp \text{ where } L = \text{Lat}(\pi_1, \ldots, \pi_k)^*.$$

Therefore the lattice-dimension of M is at most k (and may be strictly less than k). Note that given any basis $\{\pi'_1, \ldots, \pi'_k\}$ of the lattice $\text{Lat}(\pi_1, \ldots, \pi_k)$, we can write $M = \cap_{i=1}^k M(\pi'_i)$ and thereby obtain many alternate representations of the mixed lattice M.

2.2 Lattice cuts for mixed-integer sets

Given a polyhedron $P \subset \mathbb{R}^n$ and $\mathcal{M} \subseteq \mathcal{M}^k$, we define the closure of P with respect to $\mathcal{M}$ as

$$\text{Cl}(P, \mathcal{M}) = \bigcap_{M \in \mathcal{M}} \text{conv}(P \cap M).$$

Now consider a mixed-integer set

$$P^I = \{x \in \mathbb{R}^n : x \in P,\; x_i \in \mathbb{Z} \text{ for } i = 1, \ldots, l\}.$$

Any mixed-lattice $M \in \mathcal{M}^n$ leads to valid inequalities for P^I if $M \supseteq \mathbb{Z}^l \times \mathbb{R}^{n-l}$ in which case $M \in \mathcal{M}^l$ as its lattice dimension can be at most l. Furthermore, if $M = \cap_{i=1}^k M(\pi_i)$, then the last $n - l$ components of π_i need to be zero for all $i = 1, \ldots, k$, i.e., $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$. We refer to $\text{Cl}(P, \mathcal{M}^k)$ as the k-dimensional lattice closure of P.

2.3 Unimodular transformations

A linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *unimodular transformation* if it is one-to-one, invertible and maps $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Any such function has the form $f(x) = Ux$ where U is a unimodular matrix (i.e., an integral matrix with determinant ± 1). Let $M \in \mathcal{M}^1$ be a mixed-lattice with lattice dimension 1, i.e, $M = \{x \in \mathbb{R}^n : \pi^T x \in \mathbb{Z}\}$ for some nonzero $\pi \in \mathbb{Z}^n$. Then

$$f(M) = \{Ux \in \mathbb{R}^n : \pi^T x \in \mathbb{Z}\} = \{Ux \in \mathbb{R}^n : (\pi^T U^{-1})Ux \in \mathbb{Z}\} = \{x \in \mathbb{R}^n : \gamma^T x \in \mathbb{Z}\},$$

where $\gamma^T = \pi^T U^{-1}$. Therefore $f(M)$ is a mixed-lattice with lattice-dimension 1, and if $M' = \cap_{j=1}^k M_j$ where $M_j \in \mathcal{M}^1$, then $f(M') = \cap_{j=1}^k f(M_i) \in \mathcal{M}^k$. In other words, a unimodular transformation maps a mixed lattice with lattice-dimension k to a mixed-lattice with the same lattice-dimension. Affinely independent vectors stay affinely independent under invertible linear transformations and consequently the dimension of a polyhedron stays the same after a unimodular transformation. Furthermore, if $B \subset \mathbb{R}^n$ is a ball of radius r, then $f(B)$ contains a ball of radius $\bar{r} = r/\alpha$, where α is the spectral norm of U^{-1}.

If B is a basis matrix of a k-dimensional lattice L, and U is a $k \times k$ unimodular matrix, then BU is also a basis matrix of L. Conversely, given any two basis matrices B_1, B_2 of a k-dimensional lattice, there exists a $k \times k$ unimodular matrix U such that $B_1 U = B_2$. More generally if the columns of a matrix B generate a basis L, then so do the columns of BU where U is a unimodular matrix; furthermore, there exists a unimodular matrix U' such that the first $\dim(L)$ columns of BU' form a basis of L, and the remaining columns are zero. This final property can be used to show that for any k-dimensional rational linear subspace V of $\mathbb{R}^n$, there is a unimodular matrix U such that $f(x) = Ux$ maps V to the linear subspace $\mathbb{R}^k \times \{0\}^{n-k}$. See [27, Chapter 4] for details on unimodular matrices and lattices.

Given a rational lattice L, a nonzero vector in the lattice such that its Euclidean norm is the smallest among all nonzero vectors in the lattice always exists, and it is called a *shortest lattice vector*. Every lattice has a *Minkowski-reduced* basis; we do not define it formally here except to note that one of the vectors in a Minkowski-reduced basis is a shortest lattice vector. Therefore, if the columns of a matrix B generate a lattice L, then we can assume there is a unimodular matrix U such that the first $\dim(L)$ columns of BU form a basis of L, and the first column of BU is a shortest lattice vector L.

3 Well-ordered qosets

The main component of our proof technique involves establishing a dominance relationship between the members of $\mathcal{M}^k$ with regards to their effect on a given polyhedron P. Some of the results we use to this end are based on more general sets and ordering relationships among their members. In an earlier paper [19] we used a similar approach to prove that the t-branch split closure is polyhedral for any integer $t > 0$. We next review some related definitions and results from this earlier work and relate it to lattice closures of polyhedra.

For a given polyhedral set P and $M', M'' \in \mathcal{M}^k$, we say that M' *dominates* M'' *on* P if

$$\operatorname{conv}(P \cap M') \subseteq \operatorname{conv}(P \cap M'').$$

In other words, M' dominates M'' on P when all valid inequalities for P that can be derived using M'' can also be derived using M'. Consequently, given a subset of mixed lattices $\mathcal{M} \subset \mathcal{M}^k$ if $M', M'' \in \mathcal{M}$, then all valid inequalities that can be derived using $\mathcal{M}$ can also be derived using $\mathcal{M} \setminus \{M''\}$.

We say that $\mathcal{M}_f \subseteq \mathcal{M}$ is a *dominating subset for* P, if for all $M \in \mathcal{M}$, there exists a $M' \in \mathcal{M}_f$ such that M' dominates M on P. Note that for such a dominating subset $\mathcal{M}_f \subseteq \mathcal{M}$, it holds that

$$\operatorname{Cl}(P, \mathcal{M}) = \operatorname{Cl}(P, \mathcal{M}_f).$$

Furthermore, if $\mathcal{M}_f$ is finite, then it follows that $\operatorname{Cl}(P, \mathcal{M})$ is a polyhedral set.

We use this concept of domination on a given polyhedral set P to define the following binary relation $\preceq_P$ on any pair of mixed lattices $M, M' \in \mathcal{M}^k$:

$$M' \preceq_P M \qquad \text{if and only if} \qquad \operatorname{conv}(P \cap M') \subseteq \operatorname{conv}(P \cap M). \tag{4}$$

Note that the relation $\preceq_P$ defines a quasi-order on $\mathcal{M}^k$ as it is (i) reflexive (i.e., $M \preceq_P M$ for all $M \in \mathcal{M}^k$), and (ii) transitive (i.e., if $M \preceq_P M'$ and $M' \preceq_P M''$, then $M \preceq_P M''$ for all $M, M', M'' \in \mathcal{M}^k$). This relation however does not define a partial order as it is not antisymmetric (i.e., $M \preceq_P M'$ and $M' \preceq_P M$, does not necessarily imply $M = M'$ for all $M, M' \in \mathcal{M}^k$). The binary relation $\preceq_P$ together with $\mathcal{M}^k$ defines the quasi-ordered set (qoset) $(\mathcal{M}^k, \preceq_P)$. We next give an important definition related to general qosets.

Definition 1. *Given a qoset $(X, \preceq)$, we say that Y is a* dominating subset *of X if $Y \subseteq X$ and for all $x \in X$, there exists $y \in Y$ such that $y \preceq x$. Furthermore, the qoset $(X, \preceq)$ is called* fairly well-ordered *if X' has a finite dominating subset for each $X' \subseteq X$.*

We proved the next result in [19] for fairly well-ordered qosets that have a common ground set based on results from Higman [22].

Lemma 2. *If $(X, \preceq_1), \ldots, (X, \preceq_m)$ are fairly well-ordered qosets, then there is a finite set $Y \subseteq X$ such that for all $x \in X$ there exists $y \in Y$ such that $y \preceq_i x$ for all $i = 1, \ldots, m$.*

Using Lemma 2 on fairly well-ordered qosets, we next prove a result on lattice closures of polyhedra in the next section.

4 Lattice closure of bounded polyhedra

Given a collection of polyhedra $Q_1, \ldots, Q_p \subseteq \mathbb{R}^n$ and a collection of mixed lattices $\mathcal{M} \subseteq \mathcal{M}_n^k$, we define the closure of $P = \cup_{i=1}^p Q_i$ with respect to $\mathcal{M}$ as follows:

$$\operatorname{Cl}(P, \mathcal{M}) = \bigcap_{M \in \mathcal{M}} \operatorname{conv}\big(P \cap M\big)$$

Using Lemma 2, we next show that given a collection of polyhedra, if a collection of mixed lattices have a finite dominating set for each polyhedra separately, then it has a finite dominating set for the union of the polyhedra as well.

Lemma 3. *Let $Q_1, \ldots, Q_p$ be a finite collection of polyhedra in $\mathbb{R}^n$ and let $k \geq 0$. Let the qoset $(\mathcal{M}_n^k, \preceq_{Q_i})$ be fairly well-ordered for $i = 1, \ldots, p$. Then any subset of $\mathcal{M}_n^k$ has a finite dominating subset for $\cup_{i=1}^p Q_i$.*

Proof. Let $\mathcal{M}$ be an arbitrary subset of $\mathcal{M}_n^k$, and note that the qoset $(\mathcal{M}, \preceq_{Q_i})$ is fairly well-ordered for $i = 1, \ldots, p$. Applying Lemma 2 with these qosets, we see that $\mathcal{M}$ has a finite subset $\mathcal{M}_f$ such that for each M in $\mathcal{M}$, there is an $M' \in \mathcal{M}_f$ such that $M' \preceq_{Q_i} M$ for all $i = 1, \ldots, p$. In other words, $\text{conv}(Q_i \cap M') \subseteq \text{conv}(Q_i \cap M)$ for $i = 1, \ldots, p$. This, combined with the fact that

$$\text{conv}\big((\cup_{i=1}^p Q_i) \cap M\big) = \text{conv}\big(\cup_{i=1}^p \text{conv}(Q_i \cap M)\big),$$

implies that

$$\text{conv}\big((\cup_{i=1}^p Q_i) \cap M'\big) \subseteq \text{conv}\big((\cup_{i=1}^p Q_i) \cap M\big).$$

■

Lemma 4. *Let $B \subseteq \mathbb{R}^n$ be a full-dimensional ball with radius $r > 0$ and let $M \in \mathcal{M}^k$. If $M \cap B = \emptyset$, then $M = M(\pi) \cap M''$ for some $M'' \in \mathcal{M}^{k-1}$ and $\pi \in \mathbb{Z}^n$ with $\|\pi\| \leq k/r$.*

Proof. Assume that M has lattice dimension $m \leq k$. There exists integral vectors $\{\pi_1, \ldots, \pi_m\}$ such that $M = \cap_{i=1}^m M(\pi_i)$ where $\{\pi_1, \ldots, \pi_m\}$ form a Minkowski-reduced basis of $\text{Lat}(\pi_1, \ldots, \pi_m)$. Therefore, $M = L + V^\perp$ where $L = \text{Lat}(\pi_1, \ldots, \pi_m)^*$ and $V = \text{span}(\pi_1, \ldots, \pi_m)$.

Let B' be the projection of B onto V and note that B' is a ball with the same dimension as V and has the same radius as B. As $B \cap M = \emptyset$, we have $B' \cap L = \emptyset$ and consequently a result of Banaszczyk [6] (also see [7, Theorem 18.3,21.1]) implies that there exists a nonzero $v \in L^*$ such that

$$\max\{v^T x : x \in B'\} - \min\{v^T x : x \in B'\} \leq 2m.$$

If the maximum above is attained at a point $\bar{x} \in B'$, then the minimum is attained at the point

$$\bar{x} - 2r\frac{v}{||v||} \in B'$$

where r is the radius of the ball B and therefore of the ball B'. Consequently

$$v^T 2r\,\frac{v}{||v||} = 2r||v|| \leq 2m$$

and

$$||v|| \leq m/r.$$

Remember that $\{\pi_1, \ldots, \pi_m\}$ form a Minkowski-reduced basis of $L^* = \text{Lat}(\pi_1, \ldots, \pi_m)$ and therefore π_1 is a shortest nonzero vector in L^*. As $v \in L^*$, we have $\|\pi_1\| \leq \|v\| \leq m/r \leq k/r$. Setting $M'' = M(\pi_2) \cap \ldots M(\pi_m) \in \mathcal{M}^{k-1}$ completes the proof. ■

The following result was proved by Cook, Kannan and Schrijver for full-dimensional polyhedra, and extended to pointed polyhedra that are not necessarily full-dimensional in [19, Lemma 14]. We will use this technical lemma in the proof of our next result.

Lemma 5. *Let P and Q be pointed polyhedra such that $Q \subset P$. Then there exists a constant $r > 0$ such that any inequality that cuts off a vertex of Q that lies in the relative interior of P excludes a $\dim(P)-$dimensional ball $B \subset P$ of radius r.*

Lemma 6. *Let $P \subseteq \mathbb{R}^n$ be a polytope and $M' \in \mathcal{M}^k$ be a mixed-lattice. Let $M \in \mathcal{M}^k$ be such that $P \cap M \neq \emptyset$, and M is dominated by M' on all facets of P but not on P. Then there is a constant κ, that depends only on P and M', such that there is an $\tilde{M} \in \mathcal{M}^k$ that satisfies (i) $\text{aff}(P) \cap M = \text{aff}(P) \cap \tilde{M}$, (ii) $\tilde{M} = M(\pi) \cap M^2$ where $\|\pi\| \leq \kappa$ and $M^2 \in \mathcal{M}^{k-1}$, and (iii) $P \not\subset M(\pi)$.*

Proof. Let $Q = \text{conv}(P \cap M')$. If $Q = \emptyset$ then M' dominates all $M \in \mathcal{M}^k$ on P and therefore the claim holds. We therefore only consider the case when Q is nonempty; in this case Q is a polytope. As M' does not dominate M on P, $P \not\subset M$ and there exists a valid inequality $c^T x \leq \mu$ for $\text{conv}(P \cap M)$ that is not valid for Q. As Q is a polytope, $\max\{c^T x : x \in Q\}$ is bounded and has an extreme point solution $x^* \in Q$. Note that the inequality $c^T x \leq \mu$ is violated by x^*.

For any facet F of P it is true that $\text{conv}(P \cap W) \cap F = \text{conv}(F \cap W)$ for any set $W \subset \mathbb{R}^n$. Therefore, as M' dominates M on any facet F of P, we have

$$\text{conv}(P \cap M') \cap F = \text{conv}(F \cap M') \subseteq \text{conv}(F \cap M) = \text{conv}(P \cap M) \cap F.$$

Therefore, $c^T x \leq \mu$ is valid for $\text{conv}(F \cap M')$ for any facet F of P. Consequently, x^* cannot be contained in any facet of P, but must be in the relative interior of P. Applying Lemma 5 with $Q = \text{conv}(P \cap M')$, we conclude that there exists a ball B (of radius r for some fixed $r > 0$) in the relative interior of P such that

$$B \subseteq \{x \in P : c^T x > \mu\},$$

and the dimension of B is the same as that of P. Therefore $B \cap M = \emptyset$ as $c^T x \leq \mu$ is valid for $\text{conv}(P \cap M)$.

If P is full-dimensional, then $\text{aff}(P) \cap M = M$ and as the ball B is also full dimensional, Lemma 4 implies that $M = M(\pi) \cap M^2$ where $\|\pi\| \leq \kappa = k/r$ and $M^2 \in \mathcal{M}^{k-1}$. Clearly $P \not\subset M(\pi)$. We next consider the case when P is not full-dimensional.

Let $\dim(P) = t < n$. In this case there exists a unimodular transformation $\sigma(x) : \mathbb{R}^n \to \mathbb{R}^n$ – with $\sigma(x) = Ux$ for a unimodular matrix U – which maps $\text{aff}(P)$ to the affine subspace $\{x \in \mathbb{R}^n : x_{t+1} = \alpha_1, \ldots, x_n = \alpha_{n-t}\}$, where $\alpha \in \mathbb{R}^{n-t}$ is rational, and therefore $\alpha \in \frac{1}{\Delta}\mathbb{Z}^{n-t}$ for some positive integer Δ (i.e., each component of α is an integral multiple of $1/\Delta$). Note that both the unimodular matrix U and the number Δ depend on the polyhedron P. As B has the same dimension as P, we have $\sigma(B) = E \times \{\alpha\}$, where $E \subseteq \mathbb{R}^t$ contains a full-dimensional ball $\bar{B}$ of radius $\bar{r} > 0$, and $\bar{r}$ depends on r and the unimodular matrix U, see Section 2.3. Let $M^\sigma = \sigma(M)$ and $P^\sigma = \sigma(P)$; then M^σ is a mixed lattice with the same lattice dimension as M. As $\sigma(B \cap M) = \sigma(B) \cap M^\sigma = \emptyset$, we have $(\bar{B} \times \{\alpha\}) \cap M^\sigma = \emptyset$. In addition, as $P \cap M \neq \emptyset$, we have $P^\sigma \cap M^\sigma \neq \emptyset$ and therefore, there exists a point $(y_0, \alpha) \in M^\sigma$ where $y_0 \in \mathbb{R}^t$. Furthermore, as $P \not\subset M$, we have $P^\sigma \not\subset M^\sigma$.

Let the lattice dimension of M be $m \leq k$. Then there exist integral vectors $\{\gamma_1, \ldots, \gamma_m\}$ such that $M^\sigma = \cap_{i=1}^m M(\gamma_i)$. As $P^\sigma \cap M^\sigma \neq \emptyset$, we have $P^\sigma \cap M(\gamma_i) \neq \emptyset$ for all i. Let $\gamma_i = \begin{pmatrix} \mu_i \\ \nu_i \end{pmatrix}$ where $\mu_i \in \mathbb{Z}^t$ and $\nu_i \in \mathbb{Z}^{n-t}$. As $P^\sigma \not\subset M^\sigma$, we have $P^\sigma \not\subset M(\gamma_j)$ for some j. Combined with $P^\sigma \cap M(\gamma_j) \neq \emptyset$, this implies that $\mu_j \neq \mathbf{0}$. Therefore $\mathrm{Lat}(\mu_1, \ldots, \mu_m)$ is a lattice with dimension at least one. Based on the discussion in Section 2.3, we can assume that μ_1 is a shortest nonzero vector in $\mathrm{Lat}(\mu_1, \ldots, \mu_m)$. Then,

$$\begin{aligned}
(\mathbb{R}^t \times \{\alpha\}) \cap M^\sigma &= \{x \in \mathbb{R}^n : \gamma_1^T x \in \mathbb{Z}, \ldots, \gamma_m^T x \in \mathbb{Z}, x_{t+1} = \alpha_1, \ldots, x_n = \alpha_{n-t}\} \\
&= \{y \in \mathbb{R}^t : \mu_1^T y + \nu_1^T \alpha \in \mathbb{Z}, \ldots, \mu_m^T y + \nu_m^T \alpha \in \mathbb{Z}\} \times \{\alpha\} \\
&= \{y \in \mathbb{R}^t : \mu_1^T y + (\nu_1 + \tau_1)^T \alpha \in \mathbb{Z}, \ldots, \mu_m^T y + (\nu_m + \tau_m)^T \alpha \in \mathbb{Z}\} \times \{\alpha\}
\end{aligned}$$

where $\tau_i \in \Delta\mathbb{Z}^t$ for $i = 1, \ldots, m$. The last equality follows from the fact that with τ_i defined as above, $\tau_i^T \alpha$ is an integer. We choose τ_i such that $\nu_i + \tau_i = (\nu_i \bmod \Delta)$ (where we apply the mod operator componentwise). Consequently, each component of $\nu_i + \tau_i$ is contained in $\{0, \ldots, \Delta - 1\}$, for $i = 1, \ldots, m$. Letting

$$M^\Delta = \bigcap_{i=1}^m M(\tilde{\gamma}_i), \text{ where } \tilde{\gamma}_i = \begin{pmatrix} \mu_i \\ \nu_i \bmod \Delta \end{pmatrix} \text{ for } i = 1, \ldots, m,$$

we have

$$(\mathbb{R}^t \times \{\alpha\}) \cap M^\sigma = (\mathbb{R}^t \times \{\alpha\}) \cap M^\Delta,$$

and therefore $(y_0, \alpha) \in M^\Delta$. Let $\beta_i = (\nu_i \bmod \Delta)^T \alpha$. Then $(y, \alpha) \in M^\Delta$ if and only if $\mu_i^T y + \beta_i \in \mathbb{Z}$ for $i = 1, \ldots, m$, and therefore $\mu_i^T y_0 + \beta_i \in \mathbb{Z}$. Consequently, for any $y \in \mathbb{R}^t$ we have

$$\begin{aligned}
\mu_i^T y + \beta_i \in \mathbb{Z} \quad &\Leftrightarrow \quad \mu_i^T y + \beta_i - (\mu_i^T y_0 + \beta_i) \in \mathbb{Z} \\
&\Leftrightarrow \quad \mu_i^T (y - y_0) \in \mathbb{Z}
\end{aligned}$$

for $i = 1, \ldots, m$. Therefore we can write

$$\begin{aligned}
(\mathbb{R}^t \times \{\alpha\}) \cap M^\Delta &= \left(y_0 + \{y \in \mathbb{R}^t : \mu_1^T y \in \mathbb{Z}, \ldots, \mu_m^T y \in \mathbb{Z}\}\right) \times \{\alpha\} \\
&= \left(y_0 + \hat{M}\right) \times \{\alpha\}
\end{aligned}$$

where $\hat{M}$ is a mixed lattice in $\mathbb{R}^t$ with $\hat{M} = \cap_{i=1}^m M(\mu_i)$.

As $(\bar{B} \times \{\alpha\}) \cap M^\sigma = \emptyset$, we have $\bar{B} \cap (y_0 + \hat{M}) = \emptyset$. Therefore $(\bar{B} - y_0) \cap \hat{M} = \emptyset$. As $\bar{B} - y_0$ is a full-dimensional ball in $\mathbb{R}^t$ with radius $\bar{r}$, Lemma 4 implies that $\hat{M} = M(\rho) \cap M'$ where $M' \in \mathcal{M}^{m-1}$ and $\|\rho\| \leq m/\bar{r}$. But ρ lies in $\mathrm{Lat}(\mu_1, \ldots, \mu_m)$ and μ_1 is a shortest nonzero vector in this lattice, and therefore $\|\mu_1\| \leq m/\bar{r}$.

Note that $\|\nu_1 \bmod \Delta\| \leq \Delta\sqrt{n-t}$. As

$$\tilde{\gamma}_1 = \begin{pmatrix} \mu_1 \\ \nu_1 \bmod \Delta \end{pmatrix},$$

it follows that there exists a constant $\bar{\kappa}$ that depends only on P and M' such that $\|\tilde{\gamma}_1\| \leq \bar{\kappa}$. As $P^\sigma \cap M(\tilde{\gamma}_1) \neq \emptyset$ and $\mu_1 \neq \mathbf{0}$, we have $P^\sigma \not\subset M(\tilde{\gamma}_1)$. Let $\sigma^{-1}(x)$ stand for inverse transformation of $\sigma(x)$, i.e., $\sigma^{-1}(x) = U^{-1}x$. It is easy to see that

$$\sigma^{-1}(M^\Delta) = \bigcap_{i=1}^{m} M(U\tilde{\gamma}_i).$$

As $P^\sigma \not\subset M(\tilde{\gamma}_1)$ we also have $P \not\subset M(U\tilde{\gamma}_1)$. Furthermore,

$$\|U\tilde{\gamma}_1\| \leq \|U\|\|\gamma_1\| \leq \bar{\kappa}\|U\|.$$

Setting $\kappa = \bar{\kappa}\|U\|$, we see that κ depends only on P and M' and $\tilde{M} = \sigma^{-1}(M^\Delta)$ has the desired property with $\pi = U\tilde{\gamma}_1$ and $M^2 = \bigcap_{i=2}^{m} M(U\tilde{\gamma}_i)$. ∎

Lemma 7. *If $P \subset \mathbb{R}^n$ is a rational polytope and k is a positive integer, then* $\operatorname{conv}(P \cap M)$ *is a polytope for all $M \in \mathcal{M}^k$.*

Proof. Let the lattice-dimension of M be $t \leq k$ and $M = \cap_{i=1}^{t} M(\pi_i)$ where $\pi_i \in \mathbb{Z}^n$ for $i = 1, \ldots, t$. Then

$$P \cap M = \{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_t^T x \in \mathbb{Z}\}.$$

Let $D_i = \{\lfloor \alpha_i^- \rfloor, \ldots, \lceil \alpha_i^+ \rceil\}$ where $\alpha_i^- = \min\{\pi_i^T x : x \in P\}$ and $\alpha_i^+ = \max\{\pi_i^T x : x \in P\}$. Therefore,

$$P \cap M = \{x \in P : \pi_1^T x \in D_1, \ldots, \pi_t^T x \in D_t\}$$

and consequently $P \cap M$ is the finite union of bounded polyhedra implying that $\operatorname{conv}(P \cap M)$ is a bounded polyhedron. ∎

We now prove the main result of this section.

Theorem 1. *Let P be a rational polytope and let $\mathcal{M} \subseteq \mathcal{M}^k$ where k is a positive integer. Then the set $\mathcal{M}$ has a finite dominating subset for P. Consequently,* $\operatorname{Cl}(P, \mathcal{M})$ *is a polytope.*

Proof. If $P \cap M = \emptyset$ for some $M \in \mathcal{M}$, then the result trivially follows as the set $\mathcal{M}_f = \{M\}$ is a finite dominating subset of $\mathcal{M}$ for P. We therefore assume that $P \cap M \neq \emptyset$ for all $M \in \mathcal{M}$. We will prove the result by showing that $(\mathcal{M}^k, \preceq_P)$ is fairly well-ordered by induction on the dimension of P.

Let $\mathcal{M} \subseteq \mathcal{M}^k$. If $\dim(P) = 0$, then P consists of a single point. Then for any element M of $\mathcal{M}$, we have $P \cap M = P$, and the set $\mathcal{M}_f = \{M\}$ is a finite dominating subset of $\mathcal{M}$ for P. Let $\dim(P) > 0$, and assume that for all polytopes $Q \subseteq \mathbb{R}^n$ with $\dim(Q) < \dim(P)$, the qoset $(\mathcal{M}^k, \preceq_Q)$ is fairly well-ordered. Let $F_1, \ldots, F_N$ be the facets of P. As $\dim(F_i) < \dim(P)$, the qosets $(\mathcal{M}, \preceq_{F_1}), \ldots, (\mathcal{M}, \preceq_{F_N})$ are fairly well-ordered. Lemma 2 implies that there exists a finite set $\mathcal{M}_f = \{M_1, \ldots, M_p\} \subseteq \mathcal{M}$ with the following property: for all $M \in \mathcal{M}$ there exists $M_i \in \mathcal{M}_f$ such that for all $j = 1, \ldots, N$ we have

$$M_i \preceq_{F_j} M.$$

In other words, the elements of $\mathcal{M}_f$ are the dominating mixed-integer lattices in $\mathcal{M}$ for all facets of P simultaneously. Applying Lemma 6 with the polytope P and the mixed-lattice M_i we obtain a number κ_i for $i \in \{1, \ldots, p\}$, bounding the norm of the π vector described in the lemma. Let $\omega = \max_i\{\kappa_i\}$ and let $\hat{\mathcal{M}} \subseteq \mathcal{M}$ consist of elements of $\mathcal{M}$ that are not dominated on P by an element of $\mathcal{M}_f$. Then, for any $M \in \hat{\mathcal{M}}$, there exists $M' \in \mathcal{M}^{k-1}$ and $\|\pi\| \leq \omega$ such that $P \cap M = P \cap (M(\pi) \cap M')$. Picking one such π and M' for each $M \in \hat{\mathcal{M}}$, we define the following functions $g(M) = M'$, and $h(M) = \pi$ for $M \in \hat{\mathcal{M}}$.

For any fixed $\pi \in \mathbb{Z}^n$ with $\|\pi\| \leq \omega$, consider the set

$$\mathcal{M}_\pi = \{M \in \hat{\mathcal{M}} : h(M) = \pi\}.$$

If $\mathcal{M}_\pi \neq \emptyset$, then for any $M \in \mathcal{M}_\pi$, we have

$$P \cap M = (P \cap M(\pi)) \cap g(M).$$

As P is a polytope not contained in $M(\pi)$, $P \cap M(\pi)$ is the union of a finite number of polytopes, say $Q_1, \ldots, Q_l$, where $\dim(Q_i) < \dim(P)$. By the induction hypothesis, the qoset $(\mathcal{M}^{k-1}, \preceq_{Q_i})$ is fairly well-ordered for $i = 1, \ldots, l$, and therefore Lemma 3 implies that the set $\{g(M) : M \in \mathcal{M}_\pi\}$ has a finite dominating subset, say $\mathcal{M}'_\pi$, for $(P \cap M(\pi)) = \cup_{i=1}^{l} Q_i$. For each element M' of $\mathcal{M}'_\pi$ we now choose one $M \in \mathcal{M}_\pi$ such that $g(M) = M'$ to obtain a finite subset $\mathcal{M}_{\pi,f}$ of $\mathcal{M}_\pi$. Clearly, $\mathcal{M}_{\pi,f}$ is a dominating subset of $\mathcal{M}_\pi$ for P.

As each $M \in \mathcal{M}$ is either dominated by some element of $\mathcal{M}_f$ on P, or $M \in \mathcal{M}_\pi$ for some π with $\|\pi\| \leq \omega$, we have shown that

$$\mathcal{M}_f \cup \Big(\bigcup_{\|\pi\| \leq \omega} \mathcal{M}_{\pi,f} \Big)$$

is a finite dominating subset of $\mathcal{M}$ for P. ∎

5 Lattice closure of general polyhedra

In this section we extend our results to unbounded polyhedra. If a rational polyhedron P is unbounded then by the Minkowski-Weyl theorem, $P = Q + C$ where Q is a rational polytope and C is a rational polyhedral cone different from $\{0\}$, see [10]. Without loss of generality, we assume that $C = \{\sum_{i=1}^{t} \lambda_i r_i : \lambda_i \geq 0 \text{ for } i = 1, \ldots t\}$ where $r_1, \ldots, r_t$ are integral vectors in $\mathbb{R}^n$. Let

$$\bar{Q} = Q + \left\{ \sum_{i=1}^{t} \lambda_i r_i : 0 \leq \lambda_i \leq 1 \text{ for } i = 1, \ldots t \right\}, \tag{5}$$

and note that $P = \bar{Q} + C$. Let $X = \mathbb{Z}^l \times \mathbb{R}^{n-l}$ for some positive $l \le n$. By Meyer's Theorem, if $P \cap X$ is nonempty, then

$$\text{conv}(P \cap X) = \text{conv}(\bar{Q} \cap X) + C,$$

see [10]. In other words, the mixed-integer hull of P can essentially be obtained from the mixed-integer hull of $\bar{Q}$. We next observe that Meyer's result holds for general mixed-lattices and not just for $X = \mathbb{Z}^l \times \mathbb{R}^{n-l}$. It is possible to show this directly by applying Meyer's result to an extended formulation of P where the new variables are declared to be integral and then projecting down the extended formulation to the space of the original variables. Instead, we present a direct proof below.

Lemma 8. *Let $P \subseteq \mathbb{R}^n$ be an unbounded rational polyhedron, such that its Minkowski-Weyl decomposition is $P = Q + C$ and let $\bar{Q}$ be defined as in (5). For any $M \in \mathcal{M}^k$, such that $P \cap M \neq \emptyset$*

$$\text{conv}(P \cap M) = \text{conv}(\bar{Q} \cap M) + C.$$

Proof. We first show that $P \cap M = (\bar{Q} \cap M) + \bar{C}$ where

$$\bar{C} = \left\{ \sum_{i=1}^{t} \lambda_i r_i : \lambda_i \in \mathbb{Z}_+ \text{ for } i = 1, \dots t \right\}.$$

Let $x \in P \cap M$. Then, as $P = Q + C$, there exists $q \in Q$ and $\lambda_1, \dots, \lambda_t \ge 0$ such that

$$x = q + \sum_{i=1}^{t} \lambda_i r_i.$$

Thus, we can write

$$x = \left(q + \sum_{i=1}^{t} (\lambda_i - \lfloor \lambda_i \rfloor) r_i \right) + \sum_{i=1}^{t} \lfloor \lambda_i \rfloor r_i.$$

This implies that $x = \bar{q} + \bar{c}$, where $\bar{q} \in \bar{Q}$ and $\bar{c} \in \bar{C}$. As $\bar{C} \subseteq \mathbb{Z}^n \subseteq M$, we have $\bar{c} \in M$. Furthermore, as $x \in M$ and M is a mixed-integer lattice we also have $\bar{q} \in M$. Therefore, we conclude that $x \in (\bar{Q} \cap M) + \bar{C}$.

Now assume $x \in (\bar{Q} \cap M) + \bar{C}$. Then $x = \bar{q} + \bar{c}$ for some $\bar{q} \in \bar{Q} \cap M$ and $\bar{c} \in \bar{C}$. As $\bar{q} \in M$ and $\bar{C} \subseteq M$, we observe that $x \in M$. On the other hand, $\bar{Q} \subseteq P$ and $\bar{C} \subseteq C$. Since C is the recession cone of P, we conclude that $x \in P$. Therefore, $x \in P \cap M$.

Therefore $P \cap M = (\bar{Q} \cap M) + \bar{C}$. Taking convex hulls in both sides we obtain $\text{conv}(P \cap M) = \text{conv}(\bar{Q} \cap M) + \text{conv}(\bar{C})$. As $C = \text{conv}(\bar{C})$, the proof is complete. ∎

Notice that Lemma 5 implies that if $\bar{Q} \cap M = \emptyset$ then $P \cap M = \emptyset$. As $Q \subset P$, the reverse is also true and therefore we observe that $P \cap M = \emptyset$ if and only if $\bar{Q} \cap M = \emptyset$. Consequently, we have the following corollary of Lemma 5.

Corollary 9. *Let $P \in \mathbb{R}^n$ be an unbounded rational polyhedron with Minkowski-Weyl decomposition $P = Q + C$ and let $\bar{Q}$ be defined as in (5). If $\mathcal{M} \subseteq \mathcal{M}^k$ then $\text{Cl}(P, \mathcal{M}) = \emptyset$ if and only if $\text{Cl}(\bar{Q}, \mathcal{M}) = \emptyset$. Furthermore, if $\text{Cl}(P, \mathcal{M}) \neq \emptyset$ then $\text{Cl}(P, \mathcal{M}) = \text{Cl}(\bar{Q}, \mathcal{M}) + C$.*

We now prove the main result of this paper.

Theorem 2. *Let P be a rational polyhedron and let $\mathcal{M} \subseteq \mathcal{M}^k$ where k is a positive integer. Then the set $\mathcal{M}$ has a finite dominating subset for P. Consequently,* $\text{Cl}(P, \mathcal{M})$ *is a polyhedron.*

Proof. As the result holds for bounded polyhedra, we only consider the case when P is unbounded. Furthermore, if $P \cap M = \emptyset$ for some $M \in \mathcal{M}$, then $\{M\}$ is a finite dominating subset and the result follows. We therefore assume that $P \cap M \neq \emptyset$ for all $M \in \mathcal{M}$.

Assume P has the Minkowski-Weyl decomposition $P = Q + C$ and let $\bar{Q}$ be defined as in (5). As $P \cap M \neq \emptyset$ for $M \in \mathcal{M}$, it follows from Lemma 8 that $\bar{Q} \cap M \neq \emptyset$ for all $M \in \mathcal{M}$. Let M_1, M_2 be two arbitrary elements in $\mathcal{M}$. Lemma 8 implies that $\text{conv}(P \cap M_i) = \text{conv}(\bar{Q} \cap M_i) + C$ for $i = 1, 2$. If M_1 dominates M_2 on $\bar{Q}$ then

$$\text{conv}(\bar{Q} \cap M_1) \subseteq \text{conv}(\bar{Q} \cap M_2) \Rightarrow \text{conv}(P \cap M_1) \subseteq \text{conv}(P \cap M_2).$$

As $\bar{Q}$ is a polytope, Theorem 1 implies that $\mathcal{M}$ has a finite dominating subset for $\bar{Q}$, say $\mathcal{M}_f \subseteq \mathcal{M}$. Every element $M \in \mathcal{M}$ is dominated by an element of $M' \in \mathcal{M}_f$ on Q, and therefore M is dominated by M' on P. This implies that $\mathcal{M}_f$ is a finite dominating subset of $\mathcal{M}$ for P and $\text{Cl}(P, \mathcal{M}) = \text{Cl}(P, \mathcal{M}_f)$. ∎

6 Rank

Consider a mixed-integer set $P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for } i = 1, \dots, k\}$ where $P \subset \mathbb{R}^n$ is a given polyhedron and $0 \leq k \leq n$. Let $\mathcal{M} = \{M \in \mathcal{M}^k : M \supseteq \mathbb{Z}^k \times \mathbb{R}^{n-k}\}$. Any mixed-lattice $M \in \mathcal{M}$ leads to valid inequalities for P^I and the closure of P with respect to $\mathcal{M}$

$$\text{Cl}(P, \mathcal{M}) = \bigcap_{M \in \mathcal{M}} \text{conv}(P \cap M) = P^I$$

as $\mathbb{Z}^k \times \mathbb{R}^{n-k} \in \mathcal{M}$. Now consider $\mathcal{M} \cap \mathcal{M}^{k-1}$, the subset of mixed-lattices in $\mathcal{M}$ with lattice dimension at most $k-1$. In this section we will show that there exists a polyhedron P for which

$$\text{Cl}^q(P, \mathcal{M} \cap \mathcal{M}^{k-1}) \neq P^I,$$

for any finite $q > 0$. Here for $X \subseteq \mathcal{M}$, we define $\text{Cl}^1(P, X) = \text{Cl}(P, X)$ and

$$\text{Cl}^q(P, X) = \text{Cl}(\text{Cl}^{q-1}(P, X), X)$$

for $q > 1$. In other words, we will show that applying the closure operation repeatedly does not give the set P^I if one restricts the mixed lattices to have lattice dimension less than the number of integer variables in the set P^I.

6.1 The sets S and $P(h)$

Let $e_1, \ldots, e_n$ be the n unit vectors in $\mathbb{R}^n$. Let S be an n-dimensional simplex of the following form:

$$S = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq n, x_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

It is well-known that S does not contain any integer points in its interior. Furthermore, the vertices of S are $\mathbf{0}$ and $ne_1, \ldots, ne_n$, which are all integral, and all the inequalities in the definition of S above are facet-defining.

Remember that given a point $x^* \in \mathbb{R}^n$ and a hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$, the Euclidean distance of x^* from H is $|a^T x^* - b|/||a||$. Note that the point

$$p = (1/2, \ldots, 1/2) \in S \tag{6}$$

has distance $1/2$ from the facets of S defined by the nonnegativity inequalities and distance $\sqrt{n}/2$ from the facet defined by $\sum_{i=1}^{n} x_i \leq n$. For $x \in S$, let $d(x)$ denote the distance of x from the closest facet of S. More precisely,

$$d(x) = \min\{x_1, \ldots, x_n, (n - \sum_{i=1}^{n} x_i)/\sqrt{n}\}.$$

Using this notation, $d(p) = 1/2$ for all $n \geq 1$.

For any positive real number h, consider the set

$$P(h) = \text{conv}\big(S \times \{0\}\ ,\ \{(p, h)\}\big) \subset \mathbb{R}^{n+1}$$

and let $P(h)^I = P(h) \cap (\mathbb{Z}^n \times \mathbb{R})$. As p lies in the interior of S it is easy to see that for any $h > 0$

$$P(h)^I = (S \cap \mathbb{Z}^n) \times \{0\} \text{ and } \text{conv}(P(h)^I) = S \times \{0\}.$$

6.2 Lattice rank of $P(h)$

Let

$$\mathcal{M} = \{M \in \mathcal{M}_{n+1}^{n-1} : M \supseteq \mathbb{Z}^n \times \mathbb{R}\}. \tag{7}$$

We will show that $\text{Cl}^q(P(1), \mathcal{M}) \neq P(1)^I$ for any finite $q \geq 1$. We will prove this by showing that for any $q \geq 1$ there exists a point $(p, \gamma) \in P(1)$ with $\gamma > 0$ in $\text{Cl}^q(P(1), \mathcal{M})$. We will need the following two lemmas to prove this fact.

Lemma 10. *Let $x \in S$ and $h > 0$. If $d(x) \geq \gamma$, then $(x, 2\gamma h/n) \in P(h)$.*

Proof. If $\gamma = 0$ the claim holds trivially, therefore we will assume $\gamma > 0$ and consequently x is contained in the interior of S. Let $v_0 = \mathbf{0}$ and $v_i = ne_i$ for $i = 1, \ldots, n$. Then $\{v_0, \ldots, v_n\}$ is the set of vertices of S, and $x = \sum_{i=0}^{n} \beta_i v_i$ where $\sum_{i=0}^{n} \beta_i = 1$ and $\beta_i \geq 0$ for $i = 0, \ldots, n$. Clearly,

$x_i = \beta_i n$. As $d(x) \geq \gamma$, for all $i = 1, \ldots, n$ we have $x_i \geq \gamma$ and therefore $\beta_i \geq \gamma/n$. Furthermore, as $(n - \sum_{i=1}^n x_i)/\sqrt{n} \geq \gamma$, we have

$$\beta_0 = 1 - \sum_{i=1}^n \beta_i = 1 - \frac{1}{n}\sum_{i=1}^n x_i = (n - \sum_{i=1}^n x_i)/n \geq \gamma/\sqrt{n}.$$

The point $p \in S$ defined in equation (6) can be written as $p = \sum_{i=0}^n \alpha_i v_i$ where $\alpha_0 = 1/2$, $\alpha_i = 1/2n$ for $i = 1, \ldots, n$ and $\sum_{i=0}^n \alpha_i = 1$. Let $\tau = 2\gamma/n$. As $\beta_i \geq \gamma/n$ and $\alpha_i \leq 1/2$ we have $\beta_i \geq \tau\alpha_i$ for $i = 0, \ldots, n$. Then

$$x = \sum_{i=0}^n \beta_i v_i + \tau(p - \sum_{i=0}^n \alpha_i v_i) = \tau p + \sum_{i=0}^n (\beta_i - \tau\alpha_i) v_i.$$

Note that $\tau > 0$ and $\beta_i - \tau\alpha_i \geq 0$ for all i and $\tau + \sum_{i=0}^n (\beta_i - \tau\alpha_i) = 1$. In other words, x is a convex combination of p and the vertices of S. Then, using the same multipliers we see that the point

$$(x, \tau h) = \tau(p, h) + \sum_{i=0}^n (\beta_i - \tau\alpha_i)(v_i, 0)$$

is in $P(h)$ and the result follows. ∎

Lemma 11. *Let $n \geq 2$ be a fixed integer. Then for any $v \in \mathbb{R}^n \setminus \{0\}$, there exists a point $x \in S \cap (\mathbb{Z}^n + \text{span}(v))$ such that $d(x) \geq 1/2n$.*

Proof. Let $v = (v_1, \ldots, v_n)$. We can assume that $|v_n| \geq |v_i|$ for $i = 1, \ldots, n-1$ (by renumbering variables if necessary). Furthermore, as multiplying v by a nonzero scalar does not change the set $\text{span}(v)$, we can assume that $||v||_1 = 1$ and that $v_n > 0$. Then

$$\sum_{i=1}^n |v_i| = 1 \Rightarrow 1 \geq |v_n| = v_n \geq 1/n.$$

Now consider the point $\bar{x} = (1, \ldots, 1, 0)$ that lies on the facet of S defined by $x_n \geq 0$. It strictly satisfies the remaining facet-defining inequalities of S as (i) it has a distance of one from the hyperplanes $x_i = 0$ for $i = 1, \ldots, n-1$ associated with the non-negativity facets and (ii) it has a distance of $1/\sqrt{n}$ from the hyperplane associated with $\sum_i x_i \leq n$. Furthermore, as $v_n > 0$, it follows that $\bar{x} + \alpha v$ strictly lies inside S for small enough $\alpha > 0$ and also belongs to $\mathbb{Z}^n + \text{span}(v)$. For an $\alpha > 0$ such that $\bar{x} + \alpha v \in P$, the distance of $\bar{x} + \alpha v$ from the hyperplane $x_i = 0$ for $i = 1, \ldots, n-1$ equals its ith component, which equals

$$1 + \alpha v_i \geq 1 - \alpha|v_i| \geq 1 - \alpha,$$

and the distance from $\sum_{i=1}^n x_i = n$ equals

$$\frac{n - \sum_{i=1}^n (\bar{x} + \alpha v)_i}{\sqrt{n}} = \frac{1 - \alpha\sum_{i=1}^n v_i}{\sqrt{n}} \geq \frac{1 - \alpha||v||_1}{\sqrt{n}} = \frac{1 - \alpha}{\sqrt{n}}.$$

Finally the distance of $\bar{x} + \alpha v$ from the hyperplane $x_n = 0$ equals

$$\alpha v_n \geq \frac{\alpha}{n}.$$

Therefore if we set $\alpha = 1/2$, then the distance of $\bar{x} + \alpha v$ from any of the facets of S is at least

$$\min\{\frac{1}{2}, \frac{1}{2\sqrt{n}}, \frac{1}{2n}\} = \frac{1}{2n}.$$

■

The last ingredient we need for our result is the so-called *Height Lemma* [17] which shows that intersection of an arbitrary number of pyramids sharing the same base is a full-dimensional object provided that their apexes have bounded norm. In the statement below, the points $s^1, s^2, \ldots, s^n$ form the base of the pyramids and the points in U are the apexes.

Lemma 12 (Height Lemma[17])**.** *Let $s^1, s^2, \ldots, s^m \in \mathbb{R}^m$ be affinely independent points in the hyperplane $\{x \in \mathbb{R}^m : ax = b\}$ where $a \in \mathbb{R}^m \setminus \{0\}$ and $b \in \mathbb{R}$. Let $b' > b$ and $\kappa > 0$ be such that $U = \{x \in \mathbb{R}^m : ax \geq b', \|x\| \leq \kappa\}$ is non-empty. Then there exists a point x in $\bigcap_{q \in U} \operatorname{conv}(s^1, s^2, \ldots, s^m, q)$ satisfying the strict inequality $ax > b$.*

Theorem 3. *Let $P = P(1)$ and $\mathcal{M} = \{M \in \mathcal{M}^{n-1}_{n+1} : M \supseteq \mathbb{Z}^n \times \mathbb{R}\}$. Then $\operatorname{Cl}^q(P, \mathcal{M}) \neq P^I$ for any $q \geq 1$.*

Proof. Recall that $\operatorname{conv}(P^I) = S \times \{0\}$. We will show that for any $h > 0$, there is an $h' > 0$ such that $\operatorname{Cl}(P(h), \mathcal{M})$ contains $P(h')$. This implies that for any $t \geq 1$, $\operatorname{Cl}^t(P, \mathcal{M}) \supseteq P(h)$ for some $h > 0$ and the result follows.

Let $M = \cap_{i=1}^{n-1} M(\pi_i) \in \mathcal{M}$. As $M \in \mathcal{M}$, we have $\mathbb{Z}^n \times \mathbb{R} \subset M$, and therefore, $\pi_i = \begin{pmatrix} \pi'_i \\ 0 \end{pmatrix}$ where $\pi'_i \in \mathbb{Z}^n$ for $i = 1, \ldots, n-1$. As $\operatorname{span}(\pi'_1, \ldots, \pi'_{n-1})$ has dimension strictly less than n, there exists a nonzero vector $v \in \mathbb{R}^n$ such that v is orthogonal to $\pi'_1, \ldots, \pi'_{n-1}$. As $(\pi'_i)^T v = 0$ for all $i = 1, \ldots, n-1$, it follows that for all $y \in \mathbb{Z}^n$ and $\alpha \in \mathbb{R}$, the point $y + \alpha v$ satisfies $(\pi'_i)^T(y + \alpha v) \in \mathbb{Z}$ for $i = 1, \ldots, n-1$. Therefore, $(\mathbb{Z}^n + \operatorname{span}(v)) \times \mathbb{R}$ is contained in M. In addition, as $S \times \{0\} \subseteq P(h)$ we have

$$(S \cap (\mathbb{Z}^n + \operatorname{span}(v))) \times \{0\} \subseteq P(h) \cap M.$$

Therefore, by Lemma 11, there is a point $x^M \in S \cap (\mathbb{Z}^n + \operatorname{span}(v))$ such that $d(x^M) \geq 1/2n$. Then, we can use Lemma 10 (by letting γ in the Lemma to $1/2n$) to conclude that $(x^M, h/n^2) \in P(h)$. As $x^M \in S \cap (\mathbb{Z}^n + \operatorname{span}(v))$, we have $(x^M, 0) \in P(h) \cap M$, and therefore $(x^M, h/n^2) \in P(h) \cap M$. Let $p^M = (x^M, h/n^2)$. Therefore, for each $M \in \mathcal{M}$, we have constructed a point $p^M \in P(h) \cap M$ with $p^M_{n+1} = h/n^2$.

Recall that S is an integral polyhedron and $S \times \{0\} = \operatorname{conv}(P^I)$. Therefore $\operatorname{conv}(P(h) \cap M) \supseteq \operatorname{conv}(P^I)$ contains $S \times \{0\}$ as well as the point p^M. We can now apply Lemma 12 with $m = n+1$, and $s^1, \ldots, s^m$ standing for the vertices of $S \times \{0\}$, $a = e_{m+1}, b = 0$, and $b' = h/n^2$. As $p^M \in P(h)$, it is contained in a bounded set of the form U for all $M \in \mathcal{M}$. We can therefore infer that there exists a point

$$\bar{x} \in \bigcap_{M \in \mathcal{M}} \text{conv}(s^1, s^2, \ldots, s^m, p^M) \subseteq \bigcap_{M \in \mathcal{M}} \text{conv}(P(h) \cap M)$$

such that $\bar{x}_{m+1} > 0$. Note that the point $(\bar{x}_1, \ldots, \bar{x}_m)$ must be contained in the interior of S as $\bar{x} \in P(h)$. Therefore, for some $h' > 0$, the point

$$(p, h') \in \text{conv}(\{\bar{x}, s^1, \ldots, s^m\}) \subseteq \text{Cl}(P(h), \mathcal{M})$$

where p is defined in equation (6). But as the convex hull of $s^1, \ldots, s^m$ and (p, h') equals $P(h')$, we have $\text{Cl}(P(h), \mathcal{M}) \supseteq P(h')$. The result follows. ■

6.3 t-branch split cuts

In [16], Dash and Günlük show that the t-branch split closure of $P(1)$ does not give the convex hull of integer points after a finite number of iterations if $t < n$. In this section we show that their result follows from Theorem 3.

For a given mixed-integer set $P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for } i = 1, \ldots, l\}$ where $P \subset \mathbb{R}^n$ is a polyhedron, recall that a t-branch split cut is a valid inequality for $P \setminus \cup_{i=1}^t S_i$ where $S_i = \{x \in \mathbb{R}^n : \beta_i < \pi_i^T x < \beta_i + 1\}$ for some $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$ and $\beta_i \in \mathbb{Z}$, for all $i = 1, \ldots, t$. Note that

$$P \setminus \cup_{i=1}^t S_i = P \cap \left(\cap_{i=1}^t (\mathbb{R}^n \setminus S_i) \right)$$

Observe that

$$\mathbb{R}^n \setminus S_i \supset \{x \in \mathbb{R}^n \ : \ \pi_i^T x \in \mathbb{Z}\} = M(\pi_i).$$

Consequently,

$$P \setminus \cup_{i=1}^t S_i \supset P \cap \left(\cap_{i=1}^t M(\pi_i) \right) = P \cap M$$

for some mixed lattice M that contains $\mathbb{Z}^l \times \mathbb{R}^{n-l}$.

The $(n-1)$-branch split closure of $P = P(1)$ defined in the previous section is

$$\text{Cl}(P, \mathcal{T}) = \bigcap_{T \in \mathcal{T}} \text{conv}(P \setminus T)$$

where $\mathcal{T}$ is the collection of all $T = \cup_{i=1}^{n-1} S_i$ where $S_i \in \mathcal{S}^1$ for $i = 1, \ldots, n-1$. Let $\mathcal{M}$ be defined as in equation (7). As we have already observed that $P \setminus T \supset P \cap M$ for some $M \in \mathcal{M}$ we conclude that

$$\text{Cl}(P, \mathcal{T}) \supset \text{Cl}(P, \mathcal{M}).$$

Furthermore, the inclusion above also holds after applying the closure operator repeatedly, and consequently we have the following corollary to Theorem 3:

Corollary 13. *Let $P = P(1)$. Then $\text{Cl}^q(P, \mathcal{T}) \neq P^I$ for any $q \geq 1$.*

In the next section we extend this result to more general sets.

6.4 Lattice-free cuts

A set $F \subset \mathbb{R}^k$ is called a strictly lattice-free set for the integer lattice $\mathbb{Z}^k$ if $F \cap \mathbb{Z}^k = \emptyset$. For a given mixed-integer set $P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for } i = 1, \ldots, k\}$ where $P \subset \mathbb{R}^n$ is a polyhedron, clearly

$$\text{conv}(P^I) \subseteq \text{conv}(P \setminus (F \times \mathbb{R}^{n-k})) \subseteq P.$$

Consequently, starting with [5, 3], there has been a significant amount of recent research studying lattice-free sets to generate valid inequalities for mixed-integer sets. We next present a result that relates cuts from unbounded strictly lattice-free sets that contain a rational line to lattice cuts. We then observe that $P(1)$ has unbounded rank with respect to cuts from such lattice-free sets.

Proposition 14. *Let $P \subset \mathbb{R}^n$ be a polyhedron and let $F \subset \mathbb{R}^k$ be such that $F \cap \mathbb{Z}^k = \emptyset$. If the lineality space of F contains a non-zero rational vector, then $P \setminus (F \times \mathbb{R}^{n-k}) \supseteq P \cap M'$ for some mixed lattice $M' \in \mathcal{M} = \{M \in \mathcal{M}_n^{k-1} : M \supset \mathbb{Z}^k \times \mathbb{R}^{n-k}\}$.*

Proof. As the lineality space of F contains a non-zero rational vector, we can assume that there is one with integral components that are coprime. Let v be such a vector. Then the set $F = Q + \text{span}(v)$ for some $Q \subset \text{span}(v)^\perp$. Note that if $F \cap (\mathbb{Z}^k + \text{span}(v)) \neq \emptyset$, then there exists a point $p \in F$ such that $p = z + \alpha v$ for some $z \in \mathbb{Z}^k$ and $\alpha \in \mathbb{R}$. In this case, the integral point $z = (p - \alpha v) \in F$, a contradiction. Consequently, $F \cap (\mathbb{Z}^k + \text{span}(v)) = \emptyset$. Now consider a basis $\{b_1, \ldots, b_k\}$ of the lattice $\mathbb{Z}^k$ such that $b_k = v$. The projection of the lattice $\mathbb{Z}^k$ onto $\text{span}(v)^\perp$ is a lattice of dimension $k-1$ with basis $\{b'_1, \ldots, b'_{k-1}\}$ where b'_i denotes the projection of b_i onto $\text{span}(v)^\perp$. Call this lattice L. Then $\mathbb{Z}^k + \text{span}(v) = L + \text{span}(v)$ and thus $F \cap (L + \text{span}(v)) = \emptyset$. Furthermore, note that $L + \text{span}(v)$ is a mixed-lattice of lattice dimension $k-1$ that contains $\mathbb{Z}^k$ and therefore it is an element of $\mathcal{M}_k^{k-1}$. Consequently

$$P \setminus (F \times \mathbb{R}^{n-k}) \supseteq P \cap ((L + \text{span}(v) \times \mathbb{R}^{n-k}) = P \cap M'$$

where $M' \in \mathcal{M}$. ∎

Using Theorem 3 we get the following corollary to the previous result.

Corollary 15. *Let $\mathcal{L}$ be the set of all strictly lattice free sets in $\mathbb{R}^n$ that have a lineality space containing a non-zero rational vector. Let $P(1)$ be defined as in Section 6.1 and*

$$\text{Cl}(P(1), \mathcal{L}) = \bigcap_{F \in \mathcal{L}} \text{conv}(P(1) \setminus (F \times \mathbb{R})).$$

Then, $\text{Cl}^q(P(1), \mathcal{L}) \neq P(1)^I$ for any $q \geq 1$.

Note that the above result still holds when lattice-free irrational hyperplanes are included in the set $\mathcal{L}$. This is due to the fact that if H is such a hyperplane, its lineality space contains a non-zero vector v (which may be irrational) and therefore $\mathbb{R}^n \setminus H \supset \mathbb{Z}^n + \text{span}(v)$. Therefore, Lemma 11 as well as the proof of Theorem 3 still apply.

In addition, it is not hard to see that Corollary 13 is a special case of Corollary 15 as each $(n-1)$-branch split set contained in $\mathcal{T}$ is a strictly lattice-free set and has a lineality space containing a non-zero rational vector.

7 Concluding remarks

Dash, Dey and Günlük [15] studied 2-dimensional lattice cuts. In this paper, we generalized this idea and studied k-dimensional lattice cuts for any positive k. In [15], it was shown that the family of 2-dimensional lattice cuts is the same as the family of crooked cross cuts, and thus the respective closures are the same object. Therefore, our main result showing that the k-dimensional lattice closure of a rational polyhedron is a polyhedron implies the same result for crooked cross closures.

We also showed that iterating the k-dimensional lattice closure (for a particular polyhedron) finitely many times does not yield the integer hull. This result is quite strong and it implies a number of previous results. It implies a similar result for split cuts proved by Cook, Kannan and Schrijver [11], and a similar result for t-branch split cuts proved in [16].

Any full-dimensional, maximal, convex lattice-free set that is unbounded is known to be a polyhedron where the recession cone equals its lineality space and is rational [25, 9]. There is a lot of recent work on deriving valid inequalities for polyhedral mixed-integer sets of the form P^I (in the previous section) by subtracting the interiors of maximal convex lattice-free sets from P and convexifying the remaining points.

Remark 16. *Our result in the previous section implies that finitely many iterations of the closure of $P(1)$ with respect to the family of all unbounded, full-dimensional, maximal, convex lattice-free sets do not yield the integer hull of $P(1)$.*

In earlier discussions, Santanu S. Dey suggested obtaining valid inequalities for a polyhedral mixed-integer set $P^I = P \cap (\mathbb{Z}^l \times \mathbb{R}^{n-l})$ from lower dimensional maximal, convex, lattice-free sets as follows. Let $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$ for $i = 1, \ldots, k$ where $k < l$. Let $\bar{x} \in P^I$, then $\bar{z} = (\pi_1^T \bar{x}, \ldots, \pi_k^T \bar{x}) \in \mathbb{Z}^k$, and consequently $\bar{z}$ is not contained in the interior of any lattice-free set in $\mathbb{R}^k$. Let $T \subset \mathbb{R}^k$ be a maximal, convex lattice-free set, for example T can be a lattice-free triangle when $k = 2$. Any linear inequality valid for $\mathrm{conv}(P \setminus C)$, where

$$C = \{x \in \mathbb{R}^n : (\pi_1^T x, \ldots, \pi_k^T x) \in T\},$$

is valid for P^I. As $k < l$, there exists a rational vector $v \in \mathbb{R}^l \times \{0\}^{n-l}$ orthogonal to all π_j for $j = 1, \ldots, k$ and therefore C is an unbounded lattice-free set in $\mathbb{R}^n$ (with respect to the mixed-lattice $\mathbb{Z}^l \times \mathbb{R}^{n-l}$). Therefore, the remark above implies that such inequalities cannot be iterated finitely many times to obtain $\mathrm{conv}(P(1)^I)$ when $k < l$. For example, for $n = 3$ and $k = 2$, finitely many iterations of the closure of $P(1)$ with respect to triangle-inequalities do not give $\mathrm{conv}(P(1)^I)$.

Acknowledgements

We would like to thank Santanu S. Dey for many discussions we had on multi-branch and lattice-free cuts over the years.

References

[1] K. Andersen, G. Cornuéjols, and Y. Li, Split closure and intersection cuts, *Mathematical Programming* **102** (2005), 457–493.

[2] K. Andersen, Q. Louveaux, and R. Weismantel, An analysis of mixed integer linear sets based on lattice point free convex sets,*Math. Oper. Res.* **35** (2010), 233–256.

[3] K. Andersen, Q. Louveaux, R. Weismantel, and L. Wolsey, *Inequalities from two rows of a simplex tableau*, IPCO proceedings, Lecture Notes in Computer Science, vol. 4513, 2007, pp. 1–15.

[4] G. Averkov, On finitely generated closures in the theory of cutting planes, *Discrete Optimization* **9** (2012), 209–215.

[5] E. Balas, Intersection cuts - a new type of cutting planes for integer programming, *Operations Research* **19** (1971), 19–39.

[6] W. Banaszczyk, New bounds in some transference theorems in the geometry of numbers, *Mathematische Annalen* **296** (1993), 625-635.

[7] A. Barvinok, `http://www.math.lsa.umich.edu/∼barvinok/latticenotes669.pdf`.

[8] A. Basu, R. Hildebrand, and M. Köppe, The triangle closure is a polyhedron, *Mathematical Programming* **145** (2014), 19–58.

[9] A. Basu, M. Conforti, G. Cornuéjols and G. Zambelli, Maximal Lattice-free Convex Sets in Linear Subspaces, *Mathematics of Operations Research* **35** (3) (2010), 704–720.

[10] M. Conforti, G. Cornuejols and G. Zambelli, Integer Programming, Springer, New York, 2014.

[11] W. J. Cook, R. Kannan, and A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming* **47** (1990), 155–174.

[12] D. Dadush, S. S. Dey, and J. P. Vielma, The split closure of a strictly convex body, *Operations Research Letters* **39** (2011), 121–126.

[13] D. Dadush, S. S. Dey, and J. P. Vielma, On the Chvátal-Gomory closure of a compact convex set, *Mathematical Programming* **145** (2014), 327–348.

[14] S. Dash, S. S. Dey, and O. Günlük, Two dimensional lattice-free cuts and asymmetric disjunctions for mixed-integer polyhedra, *Mathematical Programming* **135** (2012), 221–254.

[15] S. Dash, S. S. Dey, and O. Günlük, On mixed-integer sets with two integer variables, *Operations Research Letters* **39** (2010), 305–309.

[16] S. Dash, O. Günlük, On t-branch split cuts for mixed-integer programs. *Mathematical Programming* **141** (2013), 191–199.

[17] S. Dash, O. Günlük, and M. Molinaro, MIR closures of polyhedral sets, *Mathematical Programming* **121**(1) (2010), 33–60.

[18] S. Dash, O. Günlük, D. A. Morán R., On the polyhedrality of cross and quadrilateral closures, *Mathematical Programming* (2016), doi:10.1007/s10107-016-0982-x.

[19] S. Dash, O. Günlük, D. A. Morán R., On the polyhedrality of closures of multi-branch split sets and other polyhedra with bounded max-facet-width (2016).

[20] A. Del Pia and R. Weismantel, On convergence in mixed integer programming, *Mathematical Programming* **135** (2012), 397–412.

[21] J. Dunkel and A.S. Schulz, The Gomory-Chvátal closure of a non-rational polytope is a rational polytope, *Mathematics of Operations Research* **38** (2013), 63–91.

[22] F. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.* **3** (1952), 326–336.

[23] A. Khinchine, A quantitative formulation of Kroneckers theory of approximation, *Izv. Acad. Nauk SSSR, Ser. Mat.* **12** (1948), 113–122 (in Russian).

[24] Y. Li and J. P. P. Richard, Cook, Kannan and Schrijver's example revisited, *Discrete Optimization* **5** (2008), 724–734.

[25] L. Lovász, Geometry of Numbers and Integer Programming, Mathematical Programming: Recent Developements and Applications, M. Iri and K. Tanabe eds., Kluwer (1989), 177-210.

[26] A. Schrijver, On cutting planes, *Annals of Discrete Mathematics* **9** (1980), 291296.

[27] A. Schrijver, Theory of Linear and Integer Programming, Wiley, New York, 1986.

[28] J. P. Vielma, A constructive characterization of the split closure of a mixed integer linear program, *Oper. Res. Lett.* **35** (2007), 29–35.