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Extension Complexity Lower Bounds for Mixed-Integer Extended Formulations

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Abstract

We prove that any mixed-integer linear extended formulation for the matching polytope of the complete graph on n vertices, with a polynomial number of constraints, requires $\Omega(\sqrt{n/\log n})$ many integer variables. By known reductions, this result extends to the traveling salesman polytope. This lower bound has various implications regarding the existence of small mixed-integer mathematical formulations of common problems in operations research. In particular, it shows that for many classic vehicle routing problems and problems involving matchings, any compact mixed-integer linear description of such a problem requires a large number of integer variables. This provides a first non-trivial lower bound on the number of integer variables needed in such settings.

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1 Introduction

Mixed-integer linear extended formulations (MILEF) are one of the most common frameworks to create mathematical models for a wide range of real-world problems. This is not only due to the extremely high expressive power of MILEFs, but also to the fact that it is a standard problem type for modern mathematical programming solvers, which have made impressive progress during the last decade in solving large-scale mixed-integer linear programs. Not surprisingly, MILEFs enjoy widespread use and popularity among practitioners. Typically, many different formulations of a problem as a MILEF exist with different pros and cons. This is nicely exemplified by the various polynomial-sized mixed-integer formulations that have been suggested for the traveling salesman problem (TSP) and related settings (see [19, 10, 18, 17] and references therein).

Whereas many mixed-integer programming formulations of NP-hard problems have been extensively studied from an empirical/computational point of view, relatively little is known about theoretical properties and limits of possible mixed-integer formulations for a given problem. We are interested in a basic quantity of MILEFs, namely the number of integer variables. Very little is known about how many integer variables are required to obtain a *compact* MILEF describing a given optimization problem, i.e., a MILEF with a polynomial number of variables and constraints. For example, it was open whether there is a compact MILEF that describes the matching polytope with a small (even constant!) number of integer variables. In particular, a compact MILEF of the matching polytope with few integer variables may have been an interesting conceptual approach to obtain an alternative way to efficiently optimize over the matching polytope. Indeed, as Lenstra [15] showed, any linear objective can be optimized over a MILEF in time polynomial in the input size of the MILEF and k^k, where k is the number of integer variables. Hence, to efficiently optimize over the matching polytope with this approach one would need $k = O({}^{\log n}/_{\log \log n})$, where n is the number of vertices of the underlying graph. Even for NP-hard problems, like TSP, lower bounds on the number of integer variables required to describe the TSP through a MILEF are missing. In particular, this left it open whether there are ways to enforce subtour elimination through a MILEF with substantially fewer integer variables as the approaches used currently, like the well-known compact formulation of Miller, Tucker and Zemlin [16], commonly known as the MTZ formulation, which is employed for many vehicle routing problems. The goal of this work is to close this gap.

The lack of bounds on the number of integer variables needed in compact MILEFs was not only due to the difficulty of understanding and analyzing potentially complex integrality constraints in an extended space, but also to an arguably very limited understanding—at least until very recently—of what can be achieved with compact linear formulations, without any integral variables. Impressive progress has been achieved in recent years on the limits of linear extended formulations [12, 20, 8, 7, 4, 6, 5, 21], building up on the seminal work of Yannakakis [23], who showed that no symmetric compact extended formulation exists for the TSP polytope. Particularly relevant for our work is a recent breakthrough by Rothvoß [21], who showed that any extended formulation of the matching polytope needs an exponential number of constraints.

As we will show, this recent progress on the extension complexity front can also be leveraged in the context of MILEFs. More precisely, combining Rothvoß' lower bound on the extension complexity of the matching polytope, with a variety of techniques from integer programming, including flatness properties of lattice-free polyhedra and disjunctive programming, we are able to prove first lower bounds for compact MILEFs for the matching and TSP polytopes. In particular our results rule out the possibility of compact MILEFs for the matching (or TSP) polytope with a small number of integer variables.

1.1 Basic notions and our results

For any set $\mathcal{F} \subseteq \mathbb{R}^d$, a mixed-integer linear extended formulation (MILEF) of $\mathcal{F}$ is a polyhedron $Q \subseteq \mathbb{R}^p$, and sets $I, J \subseteq [p]$ with $|I| = d$ (hence $p \geq d$), such that

$$\text{conv}(\mathcal{F}) = \text{proj}_{x_I}(\text{conv}(\{x \in Q : x_j \in \mathbb{Z} \text{ for all } j \in J\})), \tag{1}$$

where proj_{x_I} denotes the linear projection onto the variables $x_I = (x_i)_{i \in I}$, and conv is the convex hull operator. For brevity, we define for any positive integer $q \in \mathbb{Z}_{>0}$ and $J \subseteq [q]$ the set

$$Z_J^q \coloneqq \{x \in \mathbb{R}^q : x_j \in \mathbb{Z} \; \forall j \in J\},$$

representing the points in $\mathbb{R}^q$ whose coordinates in J are integer. The MILEF representation of $\mathcal{F}$ described in (1) can thus be written more concisely as

$$\text{conv}(\mathcal{F}) = \text{proj}_{x_I}(\text{conv}(Q \cap Z_J^p)).$$

For simplicity, we will also call the tuple $(Q \cap Z_J^p, I)$ a MILEF of $\mathcal{F}$.

Clearly, optimizing a linear function over $\mathcal{F}$ leads to the same objective value as doing so over $Q \cap Z_J^p$—when extending the linear function with zeros when lifting it from $\mathbb{R}^d$ to $\mathbb{R}^p$—and, whenever there is a unique optimal solution to the linear optimization problem over $\mathcal{F}$, this solution is simply the projection of any optimal solution of the corresponding linear optimization problem over $\text{conv}(Q \cap Z_J^p)$. Hence, a compact description of $\mathcal{F}$ as a MILEF with few integer variables can indeed be exploited for optimizing a linear function over $\mathcal{F}$, for example by using Lenstra's algorithm [15]. To refer to the *size* of a MILEF, we use the ordered pair (m, k), where m is the number of linear inequalities needed to describe Q and $k = |J|$. Analogous to the study of linear extended formulations, it is only necessary to consider the number of linear inequalities, not the number of linear equations in the description of Q since the equations can be projected out.[1]

As Rothvoß showed [21], the matching polytope of a complete graph on n vertices, denoted by K_n, has no linear extended formulation of sub-exponential size, that is, when $k = 0$, m is exponential in n. Even so, it may be possible to model the matching polytope with a MILEF using few integer variables. Following the runtime of Lenstra's algorithm, we suggest that *few* means $O({\log n}/{\log \log n})$. We show that this is not the case for the matching polytope.

Theorem 1. *Let $m(n)$ be a polynomial in n. Any mixed-integer extended formulation of the matching polytope of K_n with fewer than $m(n)$ linear inequalities must have $k = \Omega(\sqrt{n/\log n})$ integer variables.*

As Rothvoß mentions, the matching polytope of K_n is a linear projection of both the perfect matching polytope of K_{2n} [21] and the traveling salesman polytope [23] on $O(n)$ vertices, thus leading to the following corollary.

Corollary 2. *Let $m(n)$ be a polynomial in n. Any mixed-integer extended formulation the perfect matching polytope or TSP polytope of K_n with fewer than $m(n)$ linear inequalities must have $k = \Omega(\sqrt{n/\log n})$ integer variables.*

Throughout this paper, all logarithms are with respect to base 2. The operators int, rel int, and conv denote the operations of interior, relative interior, and convex hull, respectively.

[1] To be precise, equations containing only integer variables cannot be projected out, since it would lead to integrality constraints on linear forms of the remaining variables, instead of just single variables as required in (1). However, notice that there are at most k linear independent equations on the integer variables J. Since $k \leq n$, these equations can be represented by at most $2n$ many inequalities, which does not significantly effect the asymptotic estimates of m. Alternatively, one could also allow for imposing integrality on general linear forms, which we avoided for convenience.

1.2 Additional remarks and further related work

We notice that even adding a single integrality condition on simple polytopes with few facets can lead to polytopes with an exponential number of facets and vertices. A nice example for this observation is the parity polytope P, which, in d dimensions, describes the convex hull of all vectors $x \in \{0,1\}^d$ such that $\|x\|_1$ is even. This polytope is well-known to have an exponential number of facets and vertices, and it can be described as the convex hull of a hypercube with a single integrality constraint as follows: $P = \mathrm{conv}(\{x \in [0,1]^d \mid z = \frac{1}{2}\sum_{i=1}^n x_i, z \in \mathbb{Z}\})$.

Recently, there has been some work on understanding, for some classes of polytopes, how to optimize over the integer points of a polytope by adding a small number of integer variables. In particular, Bader et al. [1] introduced the notion of *affine TU dimension* of a matrix, which can be interpreted as how far a matrix is from being totally unimodular. If a polytope $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ with integral right-hand side b has affine TU-dimension k, then k integral variables suffice to described the convex hull of its integer points as a MILEF. The notion of affine TU dimension is an extension of the notion of *nearly totally unimodular* matrices, introduced by Gijswijt and Regts [9] in the context of the Integer Carathéodory Property.

Kaibel and Weltge [13] took a somewhat orthogonal approach in which they focus on the original variable space and assume integrality of all variables, with the goal of understanding how many linear inequalities are needed to describe a certain set of integer points. More precisely, they consider the set of integer points $X \subseteq \mathbb{Z}^d$ in some polyhedron $P \subseteq \mathbb{R}^d$, and then study how many facets a polytope $P' \subseteq \mathbb{R}^d$ needs to have such that $X = P' \cap \mathbb{Z}^d$. They show that even for natural polytopes that admit a compact extended formulation, like the spanning tree polytope, an exponential number of linear constraints is often necessary.

2 Proof of the main theorem

Before going into the details, we provide an outline of our proof plan.

2.1 Rough outline of our proof

On a high level we derive a contradiction as follows. We will show that the existence of a MILEF for the matching polytope of K_n with small size (m,k)—i.e., $m = O(\mathrm{poly}(n))$ and $k \leq C\sqrt{n/\log n}$ for a well-chosen constant $C > 0$—implies the existence of a *linear* extended formulation for the matching polytope on $K_{\bar{n}}$, for a well-chosen $\bar{n} < n$, with fewer inequalities than what is required by Rothvoß' lower bound on the extension complexity of the matching polytope:

Theorem 3 ([21]). *There exists a constant $c > 0$ such that for all $n \in \mathbb{Z}_{\geq 2}$, the extension complexity of the matching polytope of K_n is at least 2^{cn}.*

To obtain this contradiction, which implies our result, we proceed as follows. We will show how a MILEF for the matching polytope on K_n can be transformed into a MILEF with at least one fewer integer variable for the matching polytope of a smaller complete graph, while controlling the increase in the number of linear constraints in the new MILEF. We will then apply this result repeatedly to eliminate all integer variables and obtain a linear extended formulation for the matching polytope on a smaller complete graph.

To eliminate one integer variable in the process of going to a smaller graph, we make heavy use of the theory of lattice-free polyhedra. More precisely, we will first show that a MILEF for the matching polytope of K_n of small size implies that there is a MILEF with the same number of integer variables for a smaller number of vertices with the following property. After a judiciously chosen reparameterization of the integral variables, there is a one integer variable x_j that only takes polynomially many values. We then eliminate x_j by first considering the description of each slice defined by the different values of x_j—each such slice can be described with one less integer variable—and then taking their convex hull using disjunctive programming.

To show that an integer variable with small range can be obtained through reparameterization, we will exhibit a lattice-free polytope on which we can invoke a version of the well-known Flatness Theorem of lattice-free polyhedra stated below. A polytope $K \subseteq \mathbb{R}^d$ is said to be *lattice-free* if $\operatorname{int}(K) \cap \mathbb{Z}^d = \emptyset$.

Theorem 4 (Flatness Theorem [14][2]). *There is a function $\phi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ such that for every $d \in \mathbb{Z}_{>0}$ and every convex, closed, and full-dimensional set $K \subseteq \mathbb{R}^d$ with $\operatorname{int}(K) \cap \mathbb{Z}^d = \emptyset$, there exists a direction $v \in \mathbb{Z}^d \setminus \{0\}$ such that*

$$\sup_{x \in K} v^T x - \inf_{x \in K} v^T x < \phi(d).$$

Moreover, $\phi(d) < \tilde{c} \cdot d^{3/2}$ for some universal constant $\tilde{c} > 0$ [3].

We will now provide technical details of how we realize the above proof plan.

2.2 From the elimination of one integer variable to Theorem 1

The following theorem is our main technical ingredient, stating that a MILEF for the matching polytope implies another MILEF for the matching polytope of a smaller graph with one fewer integer variable.

Lemma 5. *Let $m, k, n \in \mathbb{Z}_{\geq 1}$ be such that there is a MILEF for the matching polytope of K_n of size (m, k) and satisfying $m < 2^{cn}$, where c is the constant from Theorem 3. Then, there is a MILEF for the matching polytope of $K_{n'}$ of size (m', k'), where*

(i) $n' \geq n - \left\lfloor \frac{\log m}{c} + 2 \right\rfloor$,
(ii) $k' = k - 1$,
(iii) $m' \leq 2m \cdot \phi(k)$,

and $\phi(k) = O(k^{3/2})$ is the function defined in Theorem 4.

Before proving Lemma 5, which we defer to Section 2.3, we first show how it implies our main result, Theorem 1.

Proof of Theorem 1. Let $\zeta, \alpha > 0$ be two constants such that the polynomial bound $m(n)$ on the number of linear constraints, as defined in the statement of the theorem, satisfies $m(n) \leq \zeta n^\alpha$. Because Theorem 1 makes a lower bound statement about k using the big-O notation, we can assume that n is larger than any fixed constant.

For the purpose of deriving a contradiction, we assume that there is a MILEF $(Q \cap Z^p_J, I)$ of the matching polytope of K_n of size (m, k) satisfying:

(i) $m \leq m(n)$, and

(ii) $k \leq C\sqrt{\frac{n}{\log n}}$, for a constant $C > 0$ to be specified later and which only depends on ζ, α, c, and $\tilde{c}$.

We recall that c and $\tilde{c}$ are the constants from Theorem 3 and Theorem 4, respectively.

We will apply Lemma 5 successively k times to eliminate all integer variables. Let $m_0 = m$ and $k_0 = k$. When applying Lemma 5 to the MILEF $(Q \cap Z^p_J, I)$, we obtain a MILEF for the matching polytope of K_{n_1}, where $n_1 \geq n - \lfloor \frac{\log m}{c} + 2 \rfloor$, of size (m_1, k_1) with $m_1 \leq 2m\phi(k_0)$ and $k_1 = k_0 - 1$. Notice that the

[2]Traditionally, this theorem is stated with the additional assumption of K being bounded. It is well-known that the stated version is equivalent. One way to see this is as follows. Consider a ball $B \subseteq K$ and look at truncated versions $K_i := K \cap [-i, i]^d$ of K, for $i \in \mathbb{Z}$ large enough such that $B \subseteq K_i$. By applying the Flatness Theorems to the K_i's, one obtains integer vectors v_i. Those vectors have bounded ℓ_2-norm; for otherwise the width of even just B will be too high. Hence, at least one vector v will appear infinitely often in the v_i's, and one can easily observe that such a v will be a flat direction for K.

conditions of Lemma 5 are indeed fulfilled because $m \leq \zeta n^{\alpha} < 2^{cn}$ for large enough n. Hence, if we apply Lemma 5 successively $i \in [k]$ times—assuming that its conditions will be fulfilled, which we will check later—then we thus obtain a MILEF for the matching polytope of K_{n_i} of size (m_i, k_i), where:

(i) $k_i = k - i$,

(ii) $m_i \leq m \cdot \prod_{j=0}^{i-1}(2\phi(k_j)) \leq m \cdot (2\tilde{c})^i \prod_{j=0}^{i-1}(k_j)^{\frac{3}{2}} = O(m(2\tilde{c})^i k^{\frac{3}{2}i})$, and

(iii) $n_i \geq n - \sum_{j=0}^{i-1} \left\lfloor \frac{\log m_i}{c} + 2 \right\rfloor$.

Hence, the last MILEF we obtain, which is actually a linear extended formulation, describes the matching polytope over a complete graph on n_k vertices, where

$$n_k \geq n - \sum_{j=0}^{k-1} \left\lfloor \frac{\log m_i}{c} + 2 \right\rfloor \ .$$

To obtain a lower bound for n_k, we first provide an upper bound for the sum in the above expression. Because we will fix the constant C later, when we use a big-O notation below, we treat C like a variable to make sure that the constant within the big-O can be chosen independently of C. The only assumption we make about C is $C \leq 1$, which allows us to simplify some expressions by replacing C by 1.

$$\begin{aligned}
\sum_{i=0}^{k-1} \left\lfloor \frac{\log m_i}{c} + 2 \right\rfloor &= O\left(\sum_{i=0}^{k-1}\left(\frac{1}{c}\left(\log m + i\log(2\tilde{c}) + \frac{3}{2} i \log k\right) + 2\right)\right) \\
&= O\left(\sum_{i=0}^{k-1}(\log m + i \log k)\right) \\
&= O\left(\sum_{i=0}^{k-1}(\log n + i \log k)\right) && \text{(using } m \leq \zeta n^{\alpha}\text{)} \\
&= O\left(k \log n + k^2 \log k\right) \\
&= C \cdot O(n). && \text{(pulling one } C \text{ out and replacing all other occurrences of } C \text{ by 1)}
\end{aligned}$$

Hence, by choosing C sufficiently small, we obtain

$$\sum_{i=0}^{k-1} \left\lfloor \frac{\log m_i}{c} + 2 \right\rfloor \leq \frac{n}{2},$$

and thus

$$n_k \geq \frac{n}{2}. \tag{2}$$

Moreover, the number of linear constraints m_k of the last MILEF can be bounded as follows:

$$m_k = O(m \cdot (2\tilde{c})^k \cdot k^{\frac{3}{2}k}) = O(m \cdot (2\tilde{c} \cdot k)^{\frac{3}{2}k}) = O\left(m \cdot \left(2\tilde{c} \cdot C\sqrt{\frac{n}{\log n}}\right)^{\frac{3}{2}C\sqrt{\frac{n}{\log n}}}\right) = 2^{o(n)}. \tag{3}$$

Notice that this implies the following, for sufficiently large n:

$$m_i < 2^{cn_i} \quad \forall\, i \in \{0, \ldots, k\}. \tag{4}$$

Indeed, $m_k < 2^{cn_k}$ for large enough n follows from (2) and (3); furthermore, for any $i \in \{0, \dots k-1\}$, we have $n_i > n_k \geq \frac{n}{2}$ and $m_i \leq O(m \cdot k^{\frac{3}{2}i}) \leq O(m \cdot k^{\frac{3}{2}k}) \leq 2^{o(n)}$, where the last inequality follows by (3). Notice that, first, (4) shows that the requirements of Lemma 5 were fulfilled each of the k times we applied the lemma. Moreover, we recall that the last MILEF we obtain is a linear extended formulation of the matching polytope of K_{n_k} with m_k linear inequalities. However, by (4) we have $m_k < 2^{cn_k}$, which violates the lower bound on the extension complexity of the matching polytope given by Theorem 3, leading to a contradiction that completes the proof. $\square$

It remains to prove Lemma 5.

2.3 Proof of Lemma 5 (eliminating one integer variable)

To prove Lemma 5, we first introduce some basic notations we use in the context of matching polytopes, and then state three auxiliary results that we use to show Lemma 5, and which we prove in Section 2.4.

Consider a complete graph on n vertices with vertex set V and edge set E. For any subset $W \subseteq V$, we denote by $E(W) \subseteq E$ all edges with both endpoints in W, and we write $P_M(W) \subseteq \mathbb{R}^{E(W)}$ for the matching polytope of the complete graph over only the vertices in W. We recall the well-known fact (see, e.g., [22]) that $P_M(W)$ can be described as follows

$$P_M(W) = \left\{ x \in \mathbb{R}^{E(W)}_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) \leq 1 & \forall v \in W \\ x(E(S)) \leq \dfrac{|S|-1}{2} & \forall S \subseteq W,\ |S| \text{ odd} \end{array} \right\},$$

where $\delta(v) \subseteq E$ indicates all edges incident with v, and for any $U \subseteq E$, we use the shorthand $x(U) := \sum_{e \in U} x(e)$. When considering MILEFs in the context of matching polytopes, we index the variables corresponding to edges simply by the corresponding edge set. For example, for $x \in P_M(V)$ and $U \subseteq E$, we denote by x_U the restriction of x to the edges in U.

We now state three auxiliary results crucial to our proof. The first one, Lemma 6, is an observation showing that facets (and also many faces) of the matching polytope are themselves lifts of a matching polytope on a smaller number of vertices. To prove Lemma 5, we will therefore present a MILEF with one fewer integer variable for an appropriate face of $P_M(V)$ which, by the lemma below, easily transforms into a MILEF of the matching polytope of a smaller graph, as desired.

Lemma 6. *Consider a complete graph on vertex set V and let $W \subseteq V$ with $|W| \geq 2$. Let $\bar{F} \subseteq \mathbb{R}^{E(W)}$ be a face of $P_M(W)$, which is defined by some hyperplane $\bar{H} \subseteq \mathbb{R}^{E(W)}$, i.e., $\bar{F} = P_M(W) \cap \bar{H}$. Let $H \subseteq \mathbb{R}^E$ be the lifted version of $\bar{H}$ into $\mathbb{R}^E$, i.e., $H = \{y \in \mathbb{R}^E : y_{E(W)} \in \bar{H}\}$, and let $F = P_M(V) \cap H$ be the corresponding face of $P_M(V)$. Then*

$$\mathrm{proj}_{x_{E(V \setminus W)}}(F) = P_M(V \setminus W).$$

To prove Lemma 5 through the above lemma, we start with a MILEF $(Q \cap Z^p_J, E)$ and create a MILEF for a particular type of face F of $P_M(V)$. More precisely, we choose a facet F of $P_M(V)$ that is not captured by Q, i.e., the inequality defining the facet F is violated by $\mathrm{proj}_{x_E}(Q)$. In other words, the MILEF $(Q \cap Z^p_J, E)$ needs the integrality conditions on the integer variables to capture F. Our next auxiliary result, Lemma 7, shows that the lifted version of such a facet F has a very structured interaction with the integral variables, which can be described in terms of a lattice-free body. This lattice-free body will allow us to invoke the Flatness Theorem to show that there is an integer direction with small width.

Lemma 7. *Let $P \subseteq \mathbb{R}^d$ be a polytope described by a MILEF $(Q \cap Z^p_J, I)$. Let $\alpha^T x \leq \beta$, with $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, be a facet-defining inequality for P, and assume that Q does not respect the lifted version of* *this inequality, i.e.,* $\exists\, \widetilde{q} \in Q$ *such that* $\alpha^T \widetilde{q}_I > \beta$*. Let* $\tilde{H} = \{x \in \mathbb{R}^p : \alpha^T x_I = \beta\}$ *be the hyperplane corresponding to the canonical lift of* $\alpha^T x = \beta$ *to the space* $\mathbb{R}^p$*. Then*

$$K = \mathrm{proj}_{x_J}(\mathrm{conv}(Q \cap \tilde{H} \cap Z_J^p))$$

is lattice-free, i.e., $\mathrm{int}(K) \cap \mathbb{Z}^{|J|} = \emptyset$*.*

The last auxiliary result we need implies that whenever a MILEF is given such that, when only looking at the integer variables, there is an integer direction with small width, then one can eliminate one integer variables at the cost of adding linear constraints to obtain a MILEF for the same polytope with one fewer integer variable. This result follows by reparameterizing the integer variables, such that the integer direction with small width corresponds to a single variable, and then using disjunctive programming to describe the convex hull of the polynomially many slices incurred by this variable.

Lemma 8. *Let* $A \in \mathbb{R}^{m \times p}$*,* $b \in \mathbb{R}^m$ *and let* $P = \{x \in \mathbb{R}^p : Ax \leq b\}$*. Suppose that* $P \subseteq \{x \in \mathbb{R}^p : \ell \leq v \cdot x_J \leq u\}$ *for some* $\ell, u \in \mathbb{Z}$*,* $\ell \leq u$*,* $J \subseteq [p]$ *and* $v \in \mathbb{Z}^{|J|}$*. Let* $I = [p] \setminus J$*. Then there exists a polyhedron* $P' \subseteq \mathbb{R}^{p'}$ *described by* m' *inequalities such that*

$$\mathrm{proj}_{x_I}\left(\mathrm{conv}(P \cap Z_J^p)\right) = \mathrm{proj}_{x_I}(\mathrm{conv}(P' \cap Z_{J'}^{p'})),$$

where $\gamma = u - \ell + 1$*,* $p' = p\gamma + p + \gamma$*,* $m' = (m+1)\gamma$ *and* $|J'| = |J| - 1$*.*

We are now ready to prove Lemma 5.

Proof of Lemma 5. Let $(Q \cap Z_J^p, E)$ be a MILEF of $P_M(V)$ satisfying the conditions of Lemma 5, i.e., its size (m, k) satisfies $m < 2^{cn}$, where c is the constant from Theorem 3, and $m, n, k \geq 1$. Hence,

$$P_M(V) = \mathrm{proj}_{x_E}(\mathrm{conv}(Q \cap Z_J^p)). \tag{5}$$

Without loss of generality we assume $n > \lfloor \frac{\log m}{c} + 2 \rfloor$, for otherwise the conditions of Lemma 5 are trivially satisfied with $n' = 0$, $k' = k - 1 \geq 0$, and $m' = 2$ since $P_M(V) = \emptyset$. Let $W \subseteq V$ be an arbitrary subset of V with $|W| = \lfloor \frac{\log m}{c} + 2 \rfloor$. Because the restriction of a matching to a smaller set of edges is also a matching, we have

$$\mathrm{proj}_{x_{E(W)}}(P_M(V)) = P_M(W) \ .$$

Thus, $(Q \cap Z_J^p, E(W))$ is a MILEF for $P_M(W)$. Because $2^{c|W|} > m$, we have

$$\mathrm{proj}_{x_{E(W)}}(Q) \supsetneq P_M(W) \ , \tag{6}$$

for otherwise Q would be an extended formulation of $P_M(W)$ with fewer inequalities than the lower bound required by Theorem 3. Hence, there is at least one facet-defining inequality $\bar{\alpha}^T x \leq \bar{\beta}$ of $P_M(W)$, where $\bar{\alpha} \in \mathbb{R}^{E(W)}$ and $\bar{\beta} \in \mathbb{R}$, that is missing in $\mathrm{proj}_{x_{E(W)}}(Q)$, i.e., there is a point $\tilde{y} \in Q$ such that $\bar{\alpha}^T \tilde{y}_{E(W)} > \bar{\beta}$. Let $\bar{H} = \{x \in \mathbb{R}^{E(W)} : \bar{\alpha}^T x = \bar{\beta}\}$, and let $\bar{F} = P_M(W) \cap \bar{H}$ be the corresponding facet of $P_M(W)$. Moreover, we denote by $F \subseteq P_M(V)$ the facet of $P_M(V)$ obtained by the canonical lift of $\bar{F}$ to $\mathbb{R}^E$; more precisely, $F = P_M(V) \cap H$, where $H = \{x \in \mathbb{R}^E : x_{E(W)} \in \bar{H}\}$ is the canonical lift of $\bar{H}$ to $\mathbb{R}^E$.

In what follows, we will show that F admits a MILEF of size $(m', k-1)$, where $m' \leq 2m\phi(k)$. The result then immediately follows by Lemma 6, which shows that F is a lift of $P_M(V \setminus W)$.

To show that there is a MILEF of F of size $(m', k-1)$, we first obtain a MILEF of F with k integer variables through a simple adaptation of the MILEF $(Q \cap Z_J^p, E)$ of $P_M(V)$. For this let $\tilde{H} = \{\tilde{x} \in \mathbb{R}^p : x_E \in H\}$ be the canonical lift of the hyperplane H into the space $\mathbb{R}^p$ in which Q lives. One can observe that the pair $(Q \cap \tilde{H} \cap Z^p_J, E)$ is a MILEF of F, due to the following. Using $F = P_M(V) \cap H$ and (5), we have $F = \text{proj}_{x_E}(\text{conv}(Q \cap Z^p_J)) \cap H$. Moreover, because H is a supporting hyperplane with respect to $P_M(V)$—since the face F of $P_M(V)$ is described by $F = P_M(V) \cap H$—and $\tilde{H}$ is the canonical lift of H, we get $F = \text{proj}_{x_E}(\text{conv}(Q \cap Z^p_J)) \cap H = \text{proj}_{x_E}(\text{conv}(Q \cap \tilde{H} \cap Z^p_J))$. Thus $(Q \cap \tilde{H} \cap Z^p_J, E)$ is indeed a MILEF of F. Lemma 7 now implies that

$$K = \text{proj}_{x_J}(\text{conv}(Q \cap \tilde{H} \cap Z^p_J))$$

is lattice-free. If, furthermore, K is full-dimensional, we can invoke the Flatness Theorem, Theorem 4, that guarantees the existence of a direction $v \in \mathbb{Z}^k \setminus \{0\}$ such that for $\ell = \lceil \inf_{x \in K} v^T x \rceil$ and $u = \lfloor \sup_{x \in K} v^T x \rfloor$ we have

$$u - \ell < \phi(k) = O(k^{\frac{3}{2}}). \tag{7}$$

Notice that if K is not full-dimensional, then it must lie on a hyperplane, which has a rational description because K is the convex hull of integer points. Hence, the normal vector of this hyperplane can be chosen to have integer coordinates, and we can choose v to be this normal vector to obtain an even stronger statement than (7), where $u = \ell$. Thus, independently of whether K is full-dimensional, statement (7) holds for a well-chosen integer vector v.

Let $\gamma = u - \ell + 1$. We can now invoke Lemma 8 with $P = Q \cap \tilde{H}$ to obtain the existence of a polytope $Q' \subseteq \mathbb{R}^{p'}$, where $p' = p\gamma + p + \gamma$, with $m' = (m+1)\gamma$ inequalities, such that there is $J' \subseteq [p']$ with $|J'| = |J| - 1$ satisfying

$$\text{proj}_{x_{[p]\setminus J}}\left(\text{conv}(Q \cap \tilde{H} \cap Z^p_J)\right) = \text{proj}_{x_{[p]\setminus J}}\left(\text{conv}(Q' \cap Z^{p'}_{J'})\right),$$

which, by projecting both sides of the above equation onto the edge-space, implies

$$\text{proj}_{x_E}(\text{conv}(Q \cap \tilde{H} \cap Z^p_J)) = \text{proj}_{x_E}(\text{conv}(Q' \cap Z^{p'}_{J'})).$$

Since $(Q \cap \tilde{H} \cap Z^p_J, E)$ is a MILEF for F, the left-hand side of the above equation (and thus also the right-hand side) are equal to F. Hence, the pair $(Q' \cap Z^{p'}_{J'}, E)$ is also a MILEF for F. It remains to observe that this latter MILEF fulfills the properties required by the statement of Lemma 5. This is indeed the case since the number of integer variables $|J'|$ satisfies $|J'| = |J| - 1 = k - 1$, and the number of constraints satisfies

$$m' = (m+1)\gamma = (m+1)(u - \ell + 1) \leq (m+1) \cdot \phi(k) \leq 2m \cdot \phi(k),$$

where the first inequality follows by (7) and integrality of the function ϕ. $\square$

2.4 Proofs of auxiliary results

It remains to show the three auxiliary results, Lemma 6, Lemma 7, and Lemma 8.

Proof of Lemma 6. Notice that the restriction of a matching on a smaller set of edges is also a matching. Hence, because furthermore $P = \text{proj}_{x_{E(V\setminus W)}}(F)$ is a projection of a face of the matching polytope, it is a $\{0,1\}$-polytope whose vertices correspond to matchings. What remains to be shown to prove the lemma, is that for every matching $M \subseteq E(V \setminus W)$, its characteristic vector χ^M is in P.

Let $\bar{x} \in \mathbb{R}^{E(W)}$ be a vertex of $\bar{F}$. Because $\bar{F}$ is a face of $P_M(E(W))$, $\bar{x}$ is also a vertex of $P_M(E(W))$ and therefore it is the characteristic vector of some matching $\bar{M} \subseteq E(W)$. Consider any matching $M \subseteq E(V \setminus W)$ and let $\widetilde{M} = M \cup \bar{M}$. Since $\widetilde{M}$ is the union of two matchings on disjoint sets of vertices, the edge set $\widetilde{M}$ is itself a matching. Furthermore, $\text{proj}_{x_{E(W)}}(\chi^{\widetilde{M}}) = \chi^{\bar{M}} \in \bar{F}$ implies that $\chi^{\widetilde{M}} \in F$. Hence

$$\chi^M = \text{proj}_{x_{E(V\setminus W)}}(\chi^{\widetilde{M}}) \in \text{proj}_{x_{E(V\setminus W)}}(F),$$

showing that the characteristic vector of any matching $M \subseteq E(V \setminus W)$ is in $\text{proj}_{x_{E(V\setminus W)}}(F)$, as desired. $\square$

Proof of Lemma 7. For the purpose of deriving a contradiction, assume $\operatorname{int}(K) \cap \mathbb{Z}^{|J|} \neq \emptyset$ and let $\bar{y} \in \operatorname{int}(K) \cap \mathbb{Z}^{|J|}$. In particular, this implies that K is full-dimensional, i.e., $\dim(K) = k$. Because $\bar{y} \in \operatorname{int} K$, we can write $\bar{y}$ as a *strict* convex combination of $\dim(K) + 1 = k + 1$ affinely independent points $\bar{y}^1, \ldots, \bar{y}^{k+1} \in K$:

$$\bar{y} = \sum_{i=1}^{k+1} \lambda_i \bar{y}^i,$$

where $\lambda_i > 0$ for $i \in [k+1]$ and $\sum_{i=1}^{k+1} \lambda_i = 1$. For $i \in [k+1]$, let $\widetilde{y}^i \in \operatorname{conv}(Q \cap \widetilde{H} \cap Z_J^p)$ be a lift of $\bar{y}^i$, i.e.,

$$\bar{y}^i = \operatorname{proj}_{x_J}(\widetilde{y}^i) \quad \forall i \in [k+1].$$

Let

$$\widetilde{y} = \sum_{i=1}^{k+1} \lambda_i \widetilde{y}^i.$$

Since $\widetilde{y} \in \operatorname{conv}(Q \cap \widetilde{H} \cap Z_J^p)$ and $\operatorname{proj}_{x_J}(\widetilde{y}) = \bar{y} \in Z_J^p$, we have

$$\widetilde{y} \in Q \cap \widetilde{H} \cap Z_J^p. \tag{8}$$

Let

$$\widetilde{H}_{\leq} = \{x \in \mathbb{R}^p : \alpha^T x_I \leq \beta\},$$

be the canonical lift of the facet-defining inequality $\alpha^T x \leq \beta$ for the polytope P.

In what follows, we show that there is a non-zero $u \in \{x \in \mathbb{R}^p : x_J = 0\}$ such that, for a small enough $\epsilon > 0$, we have $z = \widetilde{y} + \epsilon \cdot u \in Q \setminus \widetilde{H}_{\leq}$. This will lead to a contradiction because $z \in Z_J^p$, which follows from $\widetilde{y} \in Z_J^p$ (see (8)) and $u_J = 0$, and hence $z \in (Q \cap Z_J^p) \setminus \widetilde{H}_{\leq}$. Since $z \notin \widetilde{H}_{\leq}$, its projection z_I violates the constraint $\alpha^T x \leq \beta$ which is facet-defining for P. Hence, the MILEF $(Q \cap Z_J^p, I)$ does not describe P. It remains to show the existence of a $u \in \{x \in \mathbb{R}^p : x_J = 0\}$ as described above.

Let

$$\begin{aligned} U &= \{x \in \mathbb{R}^p : x_J = 0\}, \text{ and} \\ W &= \operatorname{aff}(\{\widetilde{y}^1, \ldots, \widetilde{y}^{k+1}\}). \end{aligned}$$

Since $\widetilde{y}^i \in \widetilde{H}$ for $i \in [k+1]$, we have

$$W \subseteq \widetilde{H}.$$

Moreover, $\dim(U) = p - |J| = p - k$, and $\dim(W) = k$, because even the projection of W onto the J-variables, which is $\operatorname{proj}_{x_J}(W) = \operatorname{aff}(\{\bar{y}^1, \ldots \bar{y}^{k+1}\})$, has dimension k because $\bar{y}^1, \ldots, \bar{y}^{k+1}$ have been chosen to be affinely independent. This reasoning also shows

$$\dim(W + U) = p,$$

because the space only consisting of J-variables is the orthogonal space of the linear space U, and this space is spanned by $\operatorname{proj}_{x_J}(W)$. Hence, any vector in $\mathbb{R}^p$ can be written as a sum of a vector in W and one in U.

Let $\widetilde{q} \in Q \setminus \widetilde{H}_{\leq}$ (as also defined in the statement of the lemma), and let $w \in W$ and $u \in U$ such that $\widetilde{q} = u + w$. We claim that u has the desired properties, i.e., for small enough $\epsilon > 0$, the point $z = \widetilde{y} + \epsilon \cdot u$ satisfies $z \in Q \setminus \widetilde{H}_{\leq}$.

We start by showing $z \notin \widetilde{H}_{\leq}$. Using $u = \widetilde{q} - w$ we have $z = \widetilde{y} + \epsilon \cdot (\widetilde{q} - w)$, which implies

$$\begin{aligned}
\alpha^T z_I &= \alpha^T \widetilde{y}_I + \epsilon \cdot \alpha^T \widetilde{q}_I - \epsilon \cdot \alpha^T w_I \\
&= \beta + \epsilon \cdot \alpha^T \widetilde{q}_I - \epsilon \cdot \beta && \text{(using } \alpha^T \widetilde{y} = \beta = \alpha^T w \text{ because } \widetilde{y} \in \widetilde{H} \text{ and } w \in W \subseteq \widetilde{H}\text{)} \\
&> \beta. && \text{(using } \alpha^T \widetilde{q}_I > \beta \text{ because } \widetilde{q} \notin \widetilde{H}_{\leq}\text{, and } \epsilon > 0\text{)}
\end{aligned}$$

Hence, $z \notin \widetilde{H}_{\leq}$, as desired.

To show $z \in Q$, we write z as

$$\begin{aligned}
z &= \widetilde{y} + \epsilon \cdot u = \widetilde{y} + \epsilon \cdot (\widetilde{q} - w) = (1-\epsilon) v + \epsilon \widetilde{q} \ , \text{ where} \\
v &= \frac{1}{1-\epsilon} \widetilde{y} - \frac{\epsilon}{1-\epsilon} w = \widetilde{y} + \frac{\epsilon}{1-\epsilon} (\widetilde{y} - w).
\end{aligned}$$

Hence, z is a convex combination of v and $\widetilde{q}$. In what follows we show $v \in Q$ which, together with $\widetilde{q} \in Q$, then implies $z \in Q$. Let $\widetilde{Y} = \operatorname{conv}(\{\widetilde{y}^1, \ldots, \widetilde{y}^{k+1}\})$. By construction $\widetilde{y} \in \operatorname{int}(\widetilde{Y})$. Thus, since furthermore $w \in W = \operatorname{aff}(\widetilde{Y})$ and $v = \widetilde{y} + \frac{\epsilon}{1-\epsilon}(\widetilde{y} - w)$, we have that for a small enough $\epsilon > 0$, the point v satisfies $v \in \widetilde{Y} \subseteq Q$, thus completing the proof. □

Proof of Lemma 8. We can assume that the greatest common divisor (gcd) g of the coefficients in v is equal to 1. For otherwise we can replace v by v/g and replace the bounds ℓ and u by $\lfloor \ell/g \rfloor$ and $\lceil u/g \rceil$, respectively. Let $\tau \in J$ be an arbitrary index in J. We begin by substituting the integer variables so that we may assume that $v = e^\tau \in \{0,1\}^J$ is the standard unit vector with $v_\tau = 1$. Since $v \in \mathbb{Z}^{|J|}$ has gcd 1, there exists a unimodular matrix U containing v as one of its rows (see, e.g., [11, Lemma 3.8]), i.e.,

$$U = \begin{bmatrix} v^T \\ \bar{U} \end{bmatrix}.$$

Then

$$\begin{aligned}
\operatorname{proj}_{x_I}(P \cap Z_J^p) &= \left\{ x_I : [A_I, A_J] \begin{bmatrix} x_I \\ x_J \end{bmatrix} \leq b, x_j \in \mathbb{Z} \text{ for all } j \in J \right\} \\
&= \left\{ x_I : \left[A_I, A_J U^{-1}\right] \begin{bmatrix} x_I \\ U x_J \end{bmatrix} \leq b, x_j \in \mathbb{Z} \text{ for all } j \in J \right\} \\
&= \left\{ \bar{x}_I : \left[A_I, A_J U^{-1}\right] \begin{bmatrix} \bar{x}_I \\ \bar{x}_J \end{bmatrix} \leq b, \bar{x}_j \in \mathbb{Z} \text{ for all } j \in J \right\} \\
&= \operatorname{proj}_{\bar{x}_I}(\bar{P} \cap Z_J^p),
\end{aligned}$$

where

$$\bar{P} = \left\{ x \in \mathbb{R}^p : \left[A_I, A_J U^{-1}\right] \begin{bmatrix} x_I \\ x_J \end{bmatrix} \leq b \right\}.$$

The transition above to $\bar{x}$ holds since U is unimodular. Following the calculation above, we have that $\bar{P} \subseteq \{x \in \mathbb{R}^p : \ell \leq x_\tau \leq u\}$. We continue the proof now setting $P = \bar{P}$ and $A = \bar{A}$.

We will define P' as an extended formulation for $\operatorname{conv}\left(\cup_{i=\ell}^u P \cap \{x : x_\tau = i\}\right)$. Using disjunctive programming (see [2]), we can write this convex hull as the projection of P' where

$$\begin{aligned}
P' = \Bigg\{ & (x, y^\ell, \ldots, y^u, \lambda) \in \mathbb{R}^p \times \mathbb{R}^{\gamma \cdot p} \times \mathbb{R}^\gamma : x = \sum_{i=\ell}^u y^i, \ \sum_{i=\ell}^u \lambda_i = 1, \text{ and} \\
& \qquad Ay^i \leq b\lambda_i, \ y^i_\tau = i \cdot \lambda_i, \ \lambda_i \geq 0 \text{ for } i \in \{\ell, \ldots, u\} \Bigg\}.
\end{aligned}$$

Thus

$$\operatorname{proj}_x(P') = \operatorname{conv}\left(\bigcup_{i=\ell}^{u} (P \cap \{x : x_\tau = z_i\})\right).$$

Note that P' has $p' = \gamma p + p + \gamma$ variables and $m' = \gamma(m+1)$ inequalities. Since $P \cap Z^p_J = \cup_{i=1}^{\gamma} (P \cap \{x : x_\tau = i\}) \cap Z^p_{J'}$ for $J' = J \setminus \{\tau\}$, we have

$$\begin{aligned}
\operatorname{proj}_{x_I}(\operatorname{conv}(P \cap Z^p_J)) &= \operatorname{proj}_{x_I}\left(\operatorname{conv}\left(\bigcup_{i=1}^{\gamma} P \cap \{x : x_\tau = i\} \cap Z^p_{J'}\right)\right) \\
&= \operatorname{proj}_{x_I}\left(\operatorname{conv}\left(\operatorname{conv}\left(\bigcup_{i=1}^{\gamma} P \cap \{x : x_\tau = i\}\right) \cap Z^p_{J'}\right)\right) \\
&= \operatorname{proj}_{x_I}\left(\operatorname{conv}(\operatorname{proj}_x(P') \cap Z^p_{J'})\right) \\
&= \operatorname{proj}_{x_I}\left(\operatorname{conv}(\operatorname{proj}_x(P' \cap Z^{p'}_{J'}))\right) \\
&= \operatorname{proj}_{x_I}\left(\operatorname{proj}_x(\operatorname{conv}(P' \cap Z^{p'}_{J'}))\right) \\
&= \operatorname{proj}_{x_I}(\operatorname{conv}(P' \cap Z^{p'}_{J'})) \ .
\end{aligned}$$

□

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References

[1] J. Bader, R. Hildebrand, R. Weismantel, and R. Zenklusen. Mixed integer reformulations of integer programs and the affine TU-dimension of a matrix, 2016. http://arxiv.org/abs/1508.02940.

[2] E. Balas. Disjunctive programming: Properties of the convex hull of feasible points. *Discrete Applied Mathematics*, 89(1-3):3–44, 1998.

[3] W. Banaszczyk, A. E. Litvak, A. Pajor, and S. J. Szarek. The flatness theorem for nonsymmetric convex bodies via the local theory of Banach spaces. *Mathematics of Operations Research*, 24(3):728–750, 1999.

[4] G. Braun, S. Fiorini, S. Pokutta, and D. Steurer. Approximation limits of linear programs (beyond hierarchies). In *Proceedings of 53rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 480–489, 2012.

[5] G. Braun and S. Pokutta. Common information and unique disjointness. *Algorithmica*, pages 1–33, 2016.

[6] M. Braverman and A. Moitra. An information complexity approach to extended formulations. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC)*, pages 161–170, 2013.

[7] S. Fiorini, S. Massar, S. Pokutta, H. R. Tiwary, and R. de Wolf. Linear vs. semidefinite extended formulations: Exponential separation and strong lower bounds. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC)*, pages 95–106, 2012.

[8] S. Fiorini, T. Rothvoß, and H. R. Tiwary. Extended formulations for polygons. *Discrete & Computational Geometry*, 48(3):658–668, 2012.

[9] D. Gijswijt and G. Regts. Polyhedra with the integer carathodory property. *Journal of Combinatorial Theory, Series B*, 102(1):62 – 70, 2012.

[10] L. Gouveia and S. Voß. A classification of formulations for the (time-dependent) traveling salesman problem. *European Journal of Operational Research*, 83(1):69 – 82, 1995.

[11] R. Hildebrand and M. Köppe. A new Lenstra-type algorithm for quasiconvex polynomial integer minimization with complexity. *Discrete Optimization*, 10(1):69 – 84, 2013.

[12] V. Kaibel, K. Pashkovich, and D. O. Theis. Symmetry matters for the sizes of extended formulations. In *Proceedings of the 14th Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pages 135–148, 2010.

[13] V. Kaibel and S. Weltge. Lower bounds on the sizes of integer programs without additional variables. *Mathematical Programming*, 154(1):407–425, 2015.

[14] A. Khinchin. A quantitative formulation of Kronecker's theory of approximation (in russian). *Izvestiya Akademii Nauk SSR Seriya Matematika*, 12:113–122, 1948.

[15] H. W. Lenstra, Jr. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8:538–548, 1983.

[16] C. E. Miller, A. W. Tucker, and R. A. Zemlin. Integer programming formulation of traveling salesman problems. *Journal of the ACM*, 7(4):326–329, 1960.

[17] T. Öncan, I. K. Altınel, and G. Laporte. A comparative analysis of several asymmetric traveling salesman problem formulations. *Computers & Operations Research*, 36(3):637 – 654, 2009.

[18] A.J. Orman and H.P. Williams. *A Survey of Different Integer Programming Formulations of the Travelling Salesman Problem*, volume 9 of *Advances in Computational Management Science*, chapter Optimisation, Econometric and Financial Analysis, pages 91–104. Springer, Berlin, Heidelberg, 2007.

[19] M. Padberg and T.-Y. Sung. An analytical comparison of different formulations of the travelling salesman problem. *Mathematical Programming*, 52(1):315–357, 1991.

[20] T. Rothvoß. Some 0/1 polytopes need exponential size extended formulations. *Mathematical Programming*, 142(1):255–268, 2013.

[21] T. Rothvoß. The matching polytope has exponential extension complexity. In *Proceedings of the 46th Annual Symposium on Theory of Computing (STOC)*, pages 263–272, 2014.

[22] A. Schrijver. *Combinatorial Optimization : Polyhedra and Efficiency*. Springer, Berlin, Heidelberg, 2003.

[23] M. Yannakakis. Expressing combinatorial optimization problems by linear programs. *Journal of Computer and System Sciences*, 43(3):441 – 466, 1991.