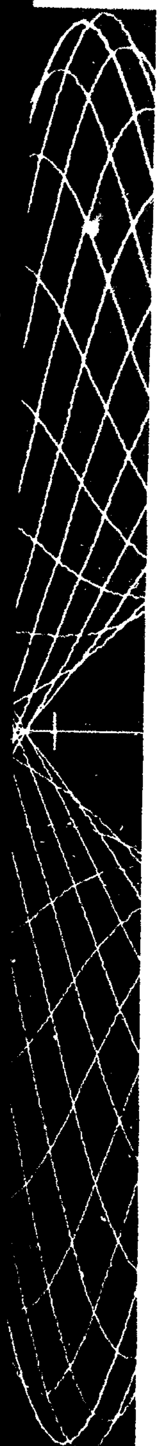

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NONLINEAR RECIPROCAL NETWORKS.

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NONLINEAR RECIPROCAL NETWORKS

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ABSTRACT: A new definition of reciprocity is given which includes the usual definitions. This is used to derive the form of the differential equations describing nonlinear reciprocal networks. The derivation clearly demonstrates the role reciprocity and Kirchhoff's laws play in producing this structure. In addition the property of reciprocity is shown to be related to the canonical transformations of Hamiltonian mechanics.

Non Linear Networks
(60) Kirchhoff's Laws

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I. INTRODUCTION

It has been recognized for some time that the equations describing electrical networks have special structures which can be exploited systematically. Indeed, all results on electrical networks must use this structure to some extent. Papers vary in the degree to which this is recognized and put in the context of the general theory. In the past ten years, the activity of viewing the equations in a general setting has increased. In part, this is due to the desire to use results in mathematics, numerical analysis, and the attempt to write general network analysis programs for the computer.

In this paper we exploit the structure of reciprocal networks. Some of our results are based on some previous results on nonlinear networks [1], [2]. However, the approach taken here is more general, extends some of the results obtained there and puts them in a more uniform context.

In Section II the structure of the differential equations describing the network will be derived directly from the reciprocity property. Reciprocity of a general n-port is given a new geometrical definition which includes the old ones and which is directly useful for derivations. These derivations are quite direct and apply equally to inductive capacitive, or resistive n-ports. The way that Kirchhoff's laws and graph theory contribute to the structure of the equations will be demonstrated. It will be shown that a generalized form of Kirchhoff's laws would not destroy the structure of the equations. In addition, the property of reciprocity is related to the canonical transformations of Hamiltonian mechanics (Lemma 4).

As in [1], the structure of the differential equations, which under some additional assumptions on the network, takes the form

$$L(i)\,\frac{di}{dt} = \frac{\partial P}{\partial i}$$

$$C(v)\,\frac{dv}{dt} = -\frac{\partial P}{\partial i},$$

is derived in Section III. This will be exploited to derive stability conditions in Section IV.

II. RECIPROCAL n-PORTS

We proceed by giving a definition of reciprocity for an n-port which will be shown to include all the usual concepts of reciprocity. The following ideas apply equally to resistive, inductive, or capacitive n-ports where the only difference is the name of the variables involved at the ports.

<u>Definition 1:</u>* An n-port is a set of n algebraic relations between 2n variables, together with a pairing of the variables so that each pair is associated with one port.

The general resistive n-port as shown in Fig. 1,

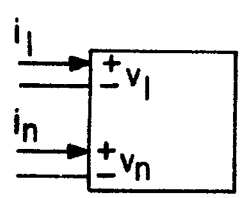

Figure 1
General Resistive n-port

where the 2n quantities are physically

*** The n in n-port is generic. Thus, two n-ports needn't have the same number of ports.**

the current i_k flowing into and out of each port, and the voltage v_k across each port. The other physical n-ports associated with electrical network theory are the inductive n-port and the capacitive n-port. In the inductive n-port, the voltage is replaced by the magnetic flux linkages φ_k. These are voltage-like quantities and obey Kirchhoff's voltage law; indeed, $\frac{d\varphi_k}{dt} = v_k$. In the capacitive n-port, the current is replaced by the electric charge q_k associated with each port. This is a current-like quantity, obeys Kirchhoff's current law, and $\frac{dq_k}{dt} = i_k$.

In what follows, for the sake of simplicity, we will designate the 2n variables at the ports as i_k, v_k, $k = 1,\ldots, n$. However, it should be understood that φ_k, q_k or possibly other quantities would do as well.

The algebraic relations representing an n-port can be viewed geometrically as an n-dimensional manifold embedded in 2n-dimensional space. The property of reciprocity of an n-port is a statement about the local nature of this manifold.

Definition 2. An n-port is said to be reciprocal if

$$\sum_{k=1}^{n} di_k \wedge dv_k = 0 \qquad (1)$$

at each point of the manifold. The symbol $\wedge$ is used to denote the exterior product of two differentials in the sense of Cartan [3].

In [2] it was demonstrated that n-ports composed of interconnections of nonlinear resistors (resistive 1-ports) satisfied the relation (1). In this paper we take (1) as a definition and the result in [2] will be shown to be a particular case. If it is possible to parametrize, the manifold representing the n-port in terms of parameters $s_1, \ldots, s_n$, i.e., some local coordinate system, then the algebraic relations can be written as

$$\begin{aligned} i_k &= i_k(s_1, \ldots, s_n) \\ v_k &= v_k(s_1, \ldots, s_n) . \end{aligned} \qquad k = 1, \ldots, n \qquad (2)$$

This is shown in Fig. 2 for $n = 3$. In this case, the

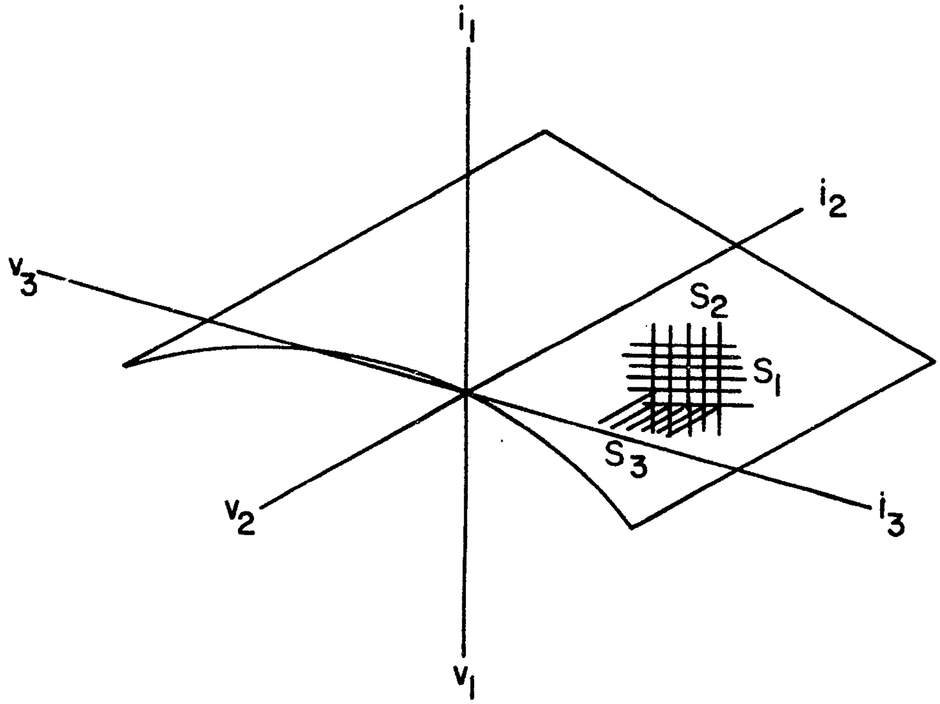

Figure 2

Reciprocal n-port Manifold

relation (1) can be expressed in terms of matrices.

Lemma 1. If (2) holds, then (1) is equivalent to

$$\left(\frac{\partial i}{\partial s}\right)^T \frac{\partial v}{\partial s} = \left(\frac{\partial v}{\partial s}\right)^T \frac{\partial i}{\partial s} . \tag{3}$$

Proof. Since i_k and v_k are scalar quantities, $di_k \wedge dv_k$ is a two-form with components

$$(di_k \wedge dv_k)_{\alpha,\beta} = \frac{\partial i_k}{\partial s_\alpha} \frac{\partial v_k}{\partial s_\beta} - \frac{\partial i_k}{\partial s_\beta} \frac{\partial v_k}{\partial s_\alpha}$$

or

$$di_k \wedge dv_k = \frac{\partial i_k}{\partial s} \left(\frac{\partial v_k}{\partial s}\right)^T - \frac{\partial v_k}{\partial s} \left(\frac{\partial i_k}{\partial s}\right)^T .$$

Therefore,

$$\sum_{k=1}^{n} di_k \wedge dv_k = \left(\frac{\partial i}{\partial s}\right)^T \frac{\partial v}{\partial s} - \left(\frac{\partial v}{\partial s}\right)^T \frac{\partial i}{\partial s} . \qquad \text{Q. E. D.}$$

Note that for $s = i$, (3) gives that $\frac{\partial v}{\partial i} = \left(\frac{\partial v}{\partial i}\right)^T$; i. e., the incremental impedance matrix is symmetrical, which is the most common definition of reciprocity.

Lemma 2. All 1-ports are reciprocal.

Proof. For a 1-port, the quantities in (3) are scalar and, hence, commute.

Definition 3. A connection n-port is one which can be obtained by taking a graph with n branches and creating a port for every branch.

Thus, inside a connection n-port, there are only nodes and the connections between nodes and ports. No two nodes are directly connected. This is illustrated in Fig. 3a and 3b.

Lemma 3. All connection n-ports are reciprocal.

Proof. Kirchhoff's laws impose the following relation among the variables at the ports. Let $v = \begin{pmatrix} \tilde{v} \\ \hat{v} \end{pmatrix}$

$i = \begin{pmatrix} \hat{i} \\ \tilde{i} \end{pmatrix}$ be the vectors of the voltage-like and current-

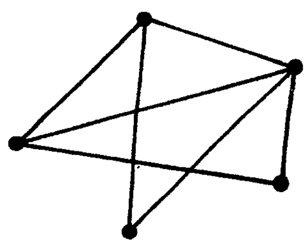

Figure 3a
Graph with 7 Branches

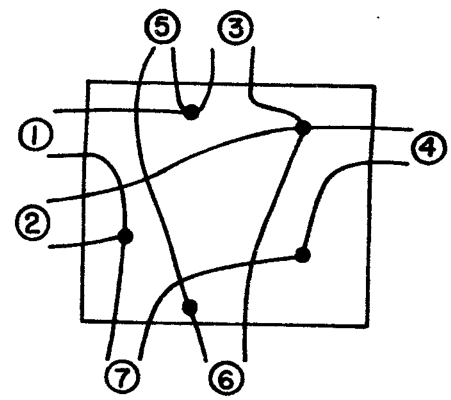

Figure 3b
7-port

like variables, respectively. Then it is well known that there exists a topological matrix E such that

$$\tilde{v} = E\,\hat{v}$$
$$\tilde{i} = -E^T\,\hat{i}$$

where E is a matrix of 0, 1, -1, and $\hat{v}$, $\hat{i}$ are respectively the vector of voltages in any maximal tree and the vector of currents in the links of that tree. Letting $s = \begin{pmatrix} \hat{i} \\ \hat{v} \end{pmatrix}$ we have

$$\frac{\partial i}{\partial s} = \begin{pmatrix} I & 0 \\ -E^T & 0 \end{pmatrix} \qquad \frac{\partial v}{\partial s} = \begin{pmatrix} 0 & E \\ 0 & I \end{pmatrix}$$

and

$$\left(\frac{\partial i}{\partial s}\right)^T \frac{\partial v}{\partial s} = \begin{pmatrix} 0 & E-E \\ 0 & 0 \end{pmatrix} = 0\ .$$

This is trivially symmetric and, hence, by Lemma 1 this n-port is reciprocal. Q. E. D.

<u>Theorem 1.</u> Any n-port obtained by the interconnection of reciprocal n-ports is reciprocal.

<u>Proof.</u> It is enough to prove this for two n-ports. The proof for an arbitrary number is similar. Since

$$\sum_{k=1}^{n} di_k \wedge dv_k = 0$$

for the first n-port and

$$\sum_{\ell=1}^{m} di_\ell \wedge dv_\ell = 0$$

for the second n-port, then clearly,

$$\sum_{k=1}^{n} di_k \wedge dv_k + \sum_{\ell=1}^{m} di_\ell \wedge dv_\ell = 0 .$$

Now decompose this sum into those ports which remain after the interconnection and those which do not --

$$\sum_{\text{remain}} + \sum_{\text{connected}} = 0 . \tag{4}$$

In addition, the interconnection could have created some new ports. Since the interconnection can be represented by a connection n-port, we have

$$\sum_{\text{connected}} + \sum_{\text{new}} = 0 . \qquad (5)$$

Adding (4) and (5) and using the fact that when two ports are connected the currents are reversed (the positive current is taken to be flowing into the top terminal of a port), we have

$$\sum_{\text{remain}} + \sum_{\text{new}} = 0 . \qquad \text{Q. E. D.}$$

This result and Lemma 1 give the result obtained in [2]:

Corollary. Any n-port constructed by the interconnection of 1-ports is reciprocal.

Obviously, the interconnection of two n-ports might be incompatible; e. g., the connection in parallel of two voltage sources of different voltage (see Fig. 4) would be meaningless. For the networks in Fig. 4 we have the relations

$$v_1 = -E_1 \qquad i_1 = s_1$$

$$v_2 = -E_1 \qquad i_2 = s_2$$

and

$$v_3 = -E_3$$

$$i_3 = s_3 .$$

The interconnection is a connection 2-port and gives $v_4 = v_5$, $i_4 = -i_5$, and $i_2 = -i_4$, $i_3 = -i_5$, $v_2 = v_4$, $v_3 = v_5$. In general, by connecting an n-port and an m-port via a connection k-port, we have

$$n+m+3k \text{ relations}$$

for $2(n+m+k)$ unknowns. An $(n+m-k)$-port results only if it is possible to use $4k$ of the above relations to eliminate $4k$ of the interior unknowns leaving $n+m-k$ relations for the $2(n+m-k)$ unknowns at the remaining

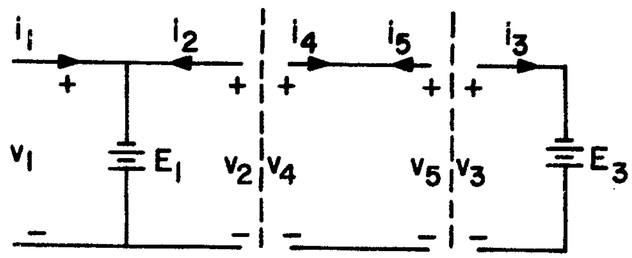

Fig. 4

Parallel Connection of

Voltage Sources

ports. If this cannot be done, we do not have an (n+m-k)-port according to Definition 1. The attention of this paper will be restricted to those interconnections which result in n-ports according to Definition 1. Note that Theorem 1 assumes that the result is an n-port in the sense stated.

Many times it is useful to consider an n-port with the parametrization $s = (i_1, \ldots, i_r, v_{r+1}, \ldots, v_n) = (\hat{i}, \hat{v})$. In discussing RLC networks, it will be necessary to see what this choice of parametrization implies.

Theorem 2. Let $v = \begin{pmatrix} \tilde{v} \\ \hat{v} \end{pmatrix}$, $i = \begin{pmatrix} \hat{i} \\ \tilde{i} \end{pmatrix}$ and $s = \begin{pmatrix} \hat{i} \\ \hat{v} \end{pmatrix}$. If a reciprocal n-port can be parametrized in terms of s, then the remaining relations for $\tilde{i}$, $\tilde{v}$ *can be obtained from a single scalar function* $F = F(\hat{i}, \hat{v})$ *as*

$$\tilde{i} = -\frac{\partial F}{\partial \hat{v}} \qquad \qquad (6)$$

$$\tilde{v} = \frac{\partial F}{\partial \hat{i}}$$

Proof.

$$\left(\frac{\partial i}{\partial s}\right)^T \frac{\partial v}{\partial s} = \begin{pmatrix} I & \left(\frac{\partial \tilde{i}}{\partial \hat{i}}\right)^T \\ 0 & \left(\frac{\partial \tilde{i}}{\partial \hat{v}}\right)^T \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \tilde{v}}{\partial \hat{i}} & \frac{\partial \tilde{v}}{\partial \hat{v}} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial \tilde{v}}{\partial \hat{i}} & \frac{\partial \tilde{v}}{\partial \hat{v}} + \left(\frac{\partial \tilde{i}}{\partial \hat{i}}\right)^T \\ 0 & \left(\frac{\partial \tilde{i}}{\partial \hat{v}}\right)^T \end{pmatrix} .$$

Thus reciprocity requires that

a) $\frac{\partial \tilde{v}}{\partial \hat{i}}, \quad \frac{\partial \tilde{i}}{\partial \hat{v}}$ are symmetric

and

b) $$\frac{\partial \tilde{i}}{\partial \hat{i}} = -\left(\frac{\partial \tilde{v}}{\partial \hat{v}}\right)^T .$$

Let

$$f(s) = \begin{pmatrix} \tilde{v}(\hat{i}, \hat{v}) \\ -\tilde{i}(\hat{i}, \hat{v}) \end{pmatrix} .$$

Then a) and b) imply that $\frac{\partial f}{\partial s}$ is symmetric and, hence, there exists a scalar function $F(s)$ such that

$$\frac{\partial F(s)}{\partial s} = f(s) . \qquad \text{Q. E. D.}$$

In [1] the function $F(\hat{i}, \hat{v})$ was denoted by $P(\hat{i}, \hat{v})$ and was called the mixed potential function. The proof in [1] that such a function exists was constructive and a recipe for constructing the mixed potential function was given. Here, the proof is nonconstructive, but more direct and has the advantage of emphasizing the role of reciprocity. In addition, the actual role that graph theory and Kirchhoff's laws play in these results can be made precise.

Definition 4. An n-port is called nonmixing if it can be parametrized by $s = \begin{pmatrix} \hat{i} \\ \hat{v} \end{pmatrix}$ and

$$\begin{aligned} \tilde{i} &= f(\hat{i}) \\ \tilde{v} &= g(\hat{v}) . \end{aligned} \qquad (7)$$

<u>Theorem 3.</u> A nonmixing n-port is reciprocal if and only if it conserves power; i. e.,

$$\sum_{k=1}^{n} i_k v_k = 0 .$$

<u>Proof.</u> Let $v = \begin{pmatrix} \tilde{v} \\ \hat{v} \end{pmatrix}$, $i = \begin{pmatrix} \hat{i} \\ \tilde{v} \end{pmatrix}$. First, suppose $i^T v = 0$. Then

$$f^T(\hat{i})\hat{v} + g^T(\hat{v})\hat{i} = 0 .$$

The second derivatives with respect to $\hat{v}$, $\hat{i}$ yield

$$\frac{\partial f(\hat{i})}{\partial \hat{i}} + \left(\frac{\partial g(\hat{v})}{\partial \hat{v}}\right)^T = 0 .$$

Let $s = \begin{pmatrix} \hat{i} \\ \hat{v} \end{pmatrix}$. Then

$$\left(\frac{\partial i}{\partial s}\right)^T \left(\frac{\partial v}{\partial s}\right) = \begin{pmatrix} I & \left(\frac{\partial f}{\partial \hat{i}}\right)^T \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{\partial g}{\partial \hat{v}} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{\partial g}{\partial \hat{v}} + \left(\frac{\partial f}{\partial \hat{i}}\right)^T \\ 0 & 0 \end{pmatrix} = 0 .$$

Hence, by Lemma 1, the n-port is reciprocal.

Suppose the n-port is reciprocal. Then according to b)

$$\frac{\partial f}{\partial \hat{i}}(\hat{i}) = -(\frac{\partial g}{\partial \hat{v}}(\hat{v}))^T .$$

This implies that both expressions are constant; i. e., $\frac{\partial f}{\partial i} \equiv -(\frac{\partial g}{\partial v})^T \equiv A$ (constant matrix). Integrating, we have $f(\hat{i}) = A\hat{i}$, $g(\hat{v}) = -A^T\hat{v}$ and

$$\begin{aligned} i^T v &= \tilde{i}^T\hat{v} + \hat{i}^T\tilde{v} \\ &= \hat{i}^T A^T \hat{v} - \hat{i}^T A^T \hat{v} = 0 . \end{aligned}$$

Q. E. D.

This theorem shows that any reciprocal network that conserves power is necessarily linear. Also, since connection n-ports are only a special class of reciprocal conservative n-ports, namely those where the matrix A contains only 0, 1, and -1 , there exists a generalization of Kirchhoff's laws which would not destroy the nature of the results obtained so far. This would allow the relation

$$\tilde{i} = A\hat{i}$$

$$\tilde{v} = -A^T\hat{v}$$

where A is an arbitrary matrix. In fact, the proof of Lemma 3 did not use that the E derived using Kirchoff's laws was composed of 0, 1, and -1 only. Lemma 3 is simply a special case of Theorem 3.

We want to reemphasize that all the results obtained so far apply to both inductive and capacitive n-ports as well as resistive n-ports. This will be used in the next section in discussing RLC networks.

Also, it should be noted that the results obtained so far allow the n-ports to vary with time or any other parameters.

An equivalent definition of reciprocity when a local parametrization exists is that the tangent space T_pM at the point p of the manifold M of a reciprocal n-port has the property that if $x, y \in T_pM$, then

$$x^TJy = 0$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The converse of this also holds; i. e., if T_pM has the above property for each $p \in M$, then M is a reciprocal n-port.

To see this, let $s(t)$ be a piece of a curve on the manifold such that $s(0) = p$. Here, s is the local coordinate patch at the point p. Then $y \in T_pM$ if and only if there exist an $s(t)$ such that

$$y = \begin{pmatrix} \frac{\partial i}{\partial s} \\ \frac{\partial v}{\partial s} \end{pmatrix} \frac{ds}{dt} \Big|_{t=0} .$$

Take two vectors $x, y \in T_pM$ and two corresponding pieces of curves $s_x(t)$, $s_y(t)$. Then

$$x^T J y = (\frac{ds_x}{dt})^T ((\frac{\partial i}{\partial s})^T, (\frac{\partial v}{\partial s})^T) J \begin{pmatrix} \frac{\partial i}{\partial s} \\ \frac{\partial v}{\partial s} \end{pmatrix} \frac{ds_y}{dt}$$

$$= (\frac{ds_x}{dt})^T \{(\frac{\partial i}{\partial s})^T \frac{\partial v}{\partial s} - (\frac{\partial v}{\partial s})^T \frac{\partial i}{\partial s}\} \frac{ds_y}{dt} = 0 .$$

Conversely, if the above expression holds for all $\frac{ds_x}{dt}$, $\frac{ds_y}{dt}$, then the matrix

$$\left(\frac{\partial i}{\partial s}\right)^T \frac{\partial v}{\partial s} - \left(\frac{\partial v}{\partial s}\right)^T \frac{\partial i}{\partial s} = 0$$

and, hence, the manifold is a reciprocal network.

Let S be a simplectic matrix; i. e., $S^T JS = J$, and transform the embedding space $\begin{pmatrix} i \\ v \end{pmatrix}$ by $\begin{pmatrix} i' \\ v' \end{pmatrix} = S\begin{pmatrix} i \\ v \end{pmatrix}$. Therefore, the manifold transforms into

$$\tilde{M} = SM .$$

Since S is simplectic, $\tilde{M}$ is reciprocal if and only if M is reciprocal. This follows from

$$x^T Jy = 0 \Leftrightarrow (Sx)^T J\ Sy .$$

We can view this transformation as a $2n$ port. In fact, the following lemma shows that the canonical transformations of Hamiltonian mechanics and a class of reciprocal n-ports are the same thing.

<u>Lemma 4.</u> Any $2n$-port which can be expressed in terms of the variables at n of the ports; i. e.,

$\begin{pmatrix} i' \\ v' \end{pmatrix} = \begin{pmatrix} i'(i, v) \\ v'(i, v) \end{pmatrix}$ is reciprocal if and only if

$\begin{pmatrix} -\frac{\partial i'}{\partial i} & \frac{\partial i'}{\partial v} \\ -\frac{\partial v'}{\partial i} & \frac{\partial v'}{\partial v} \end{pmatrix}$ is simplectic; i. e., the transformation

$\begin{pmatrix} i' \\ v' \end{pmatrix} = f(-i, v)$ is canonical.

Note the reversal in sign of the currents i due to the fact that the positive orientation for an n-port is taken as flowing into the n-port.

<u>Proof.</u> We express this $2n$-port in terms of the parametrization $s = \begin{pmatrix} i \\ v \end{pmatrix}$. Then

$$\frac{\partial}{\partial s}\begin{pmatrix} i' \\ i \end{pmatrix} = \begin{pmatrix} \frac{\partial i'}{\partial i} & \frac{\partial i'}{\partial v} \\ I & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial s}\begin{pmatrix} v' \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial v'}{\partial i} & \frac{\partial v'}{\partial v} \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} (\frac{\partial i'}{\partial i})^T & I \\ (\frac{\partial i'}{\partial v})^T & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial v'}{\partial i} & \frac{\partial v'}{\partial v} \\ 0 & I \end{pmatrix} = \begin{pmatrix} (\frac{\partial i'}{\partial i})^T \frac{\partial v'}{\partial i} & I + (\frac{\partial i'}{\partial i})^T (\frac{\partial v'}{\partial v}) \\ (\frac{\partial i'}{\partial v})^T \frac{\partial v'}{\partial i} & (\frac{\partial i'}{\partial v})^T \frac{\partial v'}{\partial v} \end{pmatrix}$$

A simplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ must satisfy $M^T J M = J$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Thus

$$A^T C = C^T A$$

$$B^T D = D^T B$$

$$-C^T B + A^T D = I \, .$$

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{\partial i'}{\partial i} & \frac{\partial i'}{\partial v} \\ -\frac{\partial v'}{\partial i} & \frac{\partial v'}{\partial v} \end{pmatrix}$. Then the above matrix becomes $\begin{pmatrix} A^T C & I - A^T D \\ -B^T C & -B^T D \end{pmatrix}$ and this is symmetric if and only if $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is simplectic.

Q. E. D.

In particular, the above lemma implies that

$$\det \begin{pmatrix} -\frac{\partial i'}{\partial i} & \frac{\partial i'}{\partial v} \\ -\frac{\partial v'}{\partial i} & \frac{\partial v'}{\partial v} \end{pmatrix} = 1 \, .$$

III. RLC NETWORKS

The structure of the differential equations describing reciprocal RLC networks can be derived directly in terms of the results of Section II. The general RLC network is simply an inductive n-port, a resistive n-port, and a capacitive n-port connected in some fashion. This is shown schematically in Fig. 5.

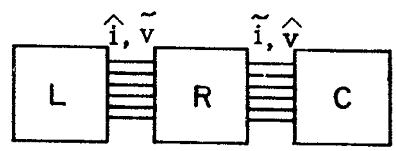

Fig. 5
General RLC Network

For simplicity, we will assume that the R n-port can be parametrized in terms of $\hat{i}$, $\hat{v}$, the L n-port in terms of $\hat{i}$ and the C n-port in terms of $\hat{v}$. Then, according to Theorem 2, there exist functions $P(\hat{i}, \hat{v})$, $\Phi(\hat{i})$ and $Q(\hat{v})$ such that

$$\tilde{v} = \frac{d}{dt} \frac{\partial}{\partial \hat{i}} \Phi(\hat{i}) = \frac{d\varphi}{dt}$$

$$\tilde{i} = \frac{d}{dt} \frac{\partial}{\partial \hat{v}} Q(\hat{v}) = \frac{dq}{dt} \qquad (8)$$

$$\tilde{v} = \frac{\partial}{\partial \hat{i}} P(\hat{i}, \hat{v})$$

$$\tilde{i} = -\frac{\partial P}{\partial \hat{v}} P(\hat{i}, \hat{v}) .$$

Obviously, other choices of parameters are possible and one would get differential equations in terms of other unknowns. The choice given above is the most common and we will restrict our attention to this.

Let

$$L(\hat{i}) \equiv \frac{\partial^2}{\partial \hat{i}^2} \Phi(\hat{i}) \qquad (9)$$

$$C(\hat{v}) \equiv \frac{\partial^2}{\partial \hat{v}^2} Q(\hat{v}) .$$

Then (8) becomes

$$L(i) \frac{di}{dt} = \frac{\partial P}{\partial i} \qquad (10)$$

$$-C(v) \frac{dv}{dt} = \frac{\partial P}{\partial v}$$

where we have dropped the $\wedge$ notation since it is no longer needed.

Equation (10) exemplifies the structure of the differential equations describing reciprocal RLC networks. The points to be emphasized about (10) are that L(i), C(v) are symmetric matrices, and the right-hand side is the gradient of a single scalar function.

To illustrate this structure, consider the network shown in Fig. 6.

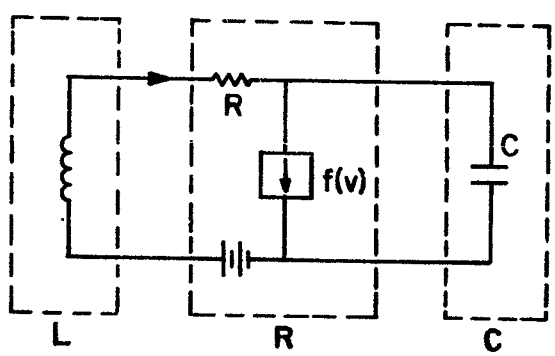

Fig. 6
RLC-Network

Here, f(v) represents the nonlinear characteristic of a tunnel diode shown in Fig. 7.

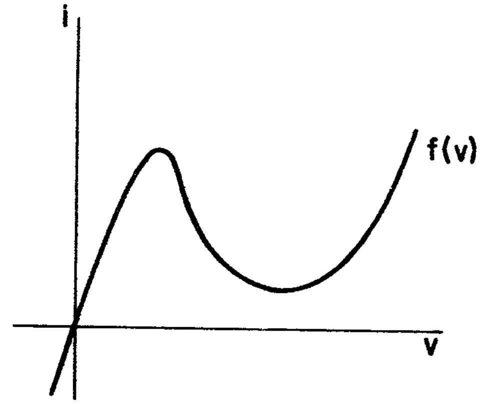

Fig. 7

Tunnel Diode Characteristic

The functions that generate the differential equations are:

$$
\begin{aligned}
P(i, v) &= Ei - vi - \frac{1}{2} Ri^2 + \int_0^v f(v')dv' \\
\Phi(i) &= \frac{1}{2} Li^2 \\
Q(v) &= \frac{1}{2} Cv^2 .
\end{aligned}
\qquad (11)
$$

Using these in (9) and (10), we have the differential equations

$$L \frac{di}{dt} = E - v - Ri$$

$$-C \frac{dv}{dt} = -i + f(v) \; . \tag{12}$$

IV. STABILITY OF THE EQUILIBRIUM SET

To motivate the type of stability of interest in some nonlinear networks, consider the network in Fig. 6. From (12), the equilibrium equations are

$$E - v - Ri = 0$$

$$i - f(v) = 0 .$$

These algebraic relations are plotted in Fig. 8.

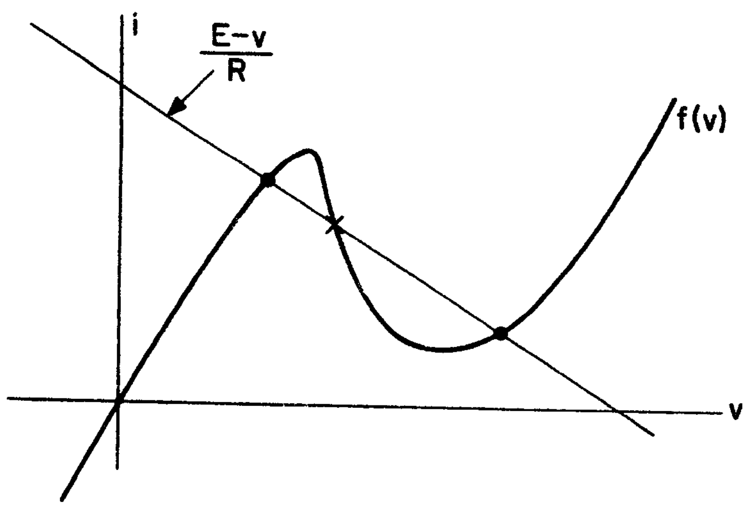

Fig. 8 Equilibrium Solutions

For some choices of E and R , as in Fig. 7, there is a multiple of equilibrium solutions. In some cases the one marked x is unstable and the ones marked • are stable. One of the applications of this network is to use it as a switching circuit where if the equilibrium resides at the leftmost stable equilibrium point, the circuit is considered to be in the "0" state; if it resides at the rightmost stable equilibrium point, it is considered to be in the "1" state. It is then necessary in applications to be able to switch the network from one state to the other in a reliable manner. To rule out the possibility that during the switching process, the network may go into an oscillation, we want to find conditions on the parameters of the network which guarantee that the equilibrium set is globally asymptotically stable. In other words, no matter what initial condition given to the equations (12), the resulting solution approaches <u>one</u> of the equilibrium points as $t \to \infty$. If this happens, the equilibrium set is said to be completely stable. In particular, this condition rules out the existence of a limit cycle.

One of the more useful results for proving complete stability in networks is a recent result by LaSalle [4]. Consider a system of autonomous differential equations

$$\dot{x} = f(x) \tag{13}$$

where x is an n-vector. A scalar function $V(x)$ is said to be a Liapunov function on R^n (in the sense of LaSalle) if $\dot{V} = (\frac{\partial V}{\partial x})^T f(x)$ does not change sign on R^n. Let

$$E = \{x;\ \dot{V}(x) = 0,\quad x \in R^n\}$$

and let M be the largest invariant set contained in E.

<u>Theorem (LaSalle)</u>. If V is a Liapunov function on R^n, then each bounded solution $x(t)$ of (13) approaches the set M as $t \to \infty$.

This theorem separates the usual Liapunov results into two parts. It allows us to search for a Liapunov function whose time derivative along solutions has only one sign, and leaves boundedness as a separate issue. In electrical networks, any realistic model of a network would have dissipation at large values of current or voltage and usually this can be used to obtain boundedness. Thus, instead of trying for two properties of $V(x)$, we need only try for $\dot{V}(x) \leq 0$, and the search for Liapunov functions is made considerably easier.

To see how LaSalle's theorem can be used in conjunction with the form of the equations for electrical networks (10) consider

$$\dot{P} = \left(\frac{\partial P}{\partial i}\right)^T \frac{di}{dt} + \left(\frac{\delta P}{\partial v}\right)^T \frac{dv}{dt}$$

$$= \left(\frac{di}{dt}\right)^T L(i) \frac{di}{dt} - \left(\frac{dv}{dt}\right)^T C(v) \frac{dv}{dt} .$$

Thus, $\dot{P}$ is a quadratic form in $\frac{di}{dt}$, $\frac{dv}{dt}$. If the network is only an RC network, then there is no i or $L(i)$ and

$$\dot{P} = -\left(\frac{dv}{dt}\right)^T C(v) \frac{dv}{dt} .$$

If $C(v)$ is positive definite, as is the case usually, then

$\dot{P} \leq 0$.

In fact, the set E is the set where $\frac{dv}{dt} = 0$ and, hence, LaSalle's theorem gives us the result that if v(t) is bounded, then v(t) approaches the equilibrium set $(\frac{dv}{dt} = 0)$ as $t \to \infty$.

Generally, if we can find a Liapunov function where $\dot{V}$ is a quadratic form in $\frac{dx}{dt}$, then the set M will be the set of equilibrium points. A family of functions for which $\dot{V}$ is a quadratic in $\frac{dx}{dt}$ is

$$V(x) = \lambda P(x) + \frac{1}{2} (\frac{\partial P}{\partial x})^T M \frac{\partial P}{\partial x}$$

where λ is a scalar constant and M a constant symmetric matrix. It is easily checked that

$$\dot{V}(x) = - (\frac{dx}{dt})^T (\lambda J + P_{xx} MJ) \frac{dx}{dt}$$

where $J = \begin{pmatrix} -L & 0 \\ 0 & C \end{pmatrix}$, $P_{xx} = \frac{\partial^2 P}{\partial x^2}$. Thus, the problem of finding a Liapunov function in the sense of LaSalle can be solved if we can find λ and M such that

$$(\lambda J + P_{xx} M J)_s$$

is at least semidefinite. Here, $(\ )_s$ denotes the symmetric part of a matrix. Of course, there are other more complicated forms which would give a quadratic in $\frac{dx}{dt}$, but the above form seems to be the most natural to investigate first.

The following theorem from [1] gives stability conditions for an electrical network.

<u>Theorem 4.</u> If $P(i, v) = -\frac{1}{2} i^T Ai + B(v) + i^T \gamma v$ plus linear terms, where A, γ are constant matrices, and

$$\| L^{1/2}(i)\, A^{-1} \gamma\, C^{-1/2}(v) \| < 1 . \qquad (14)$$

Then each bounded solution of (10) approaches the equilibrium set as $t \to \infty$.

Proof. Choose $\lambda = 1$, $M = \begin{pmatrix} 2A^{-1} & 0 \\ 0 & 0 \end{pmatrix}$.

Verify that if (14) holds, then

$$(\lambda J + P_{xx} MJ)_s < 0 \,, \tag{15}$$

and use LaSalle's theorem. Q. E. D.

As an example, this theorem can be applied to the network of Figure 6. Since

$$P(i, v) = Ei - vi - \frac{1}{2} Ri^2 + \int_0^v f(v')dv'$$

has the form prescribed and $L = L$ and $C = C$, then (14) becomes

$$\frac{L}{R^2 C} < 1 \,.$$

We note that this condition is independent of the nonlinearity $f(v)$. In fact, Theorem 4 gives a condition on the linear part of the network (assuming that L, C are constant), which is sufficient for stability.

There are other stability theorems which can be obtained by making restrictions on the type of nonlinearity. Some of these are given in [1] and [5].

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