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A Law of Large Numbers and Functional Central Limit Theorem for Generalized Semi-Markov Processes

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Abstract

The generalized semi-Markov process (GSMP) is the usual model for the underlying stochastic process of a complex discrete-event stochastic system. Strong laws of large numbers (SLLNs) and functional central limit theorems (FCLTs) give basic conditions under which such processes exhibit stable long run behavior. These limit theorems also provide approximations for cumulative-reward distributions, confidence intervals for statistical estimators, and efficiency criteria for simulation algorithms. We prove an SLLN and FCLT for finite state GSMPs under significantly weaker conditions on the moments of the clock-setting distributions than have previously been imposed. As part of our analysis, we use Lyapunov-function arguments to show that finite moments for new clock readings imply finite moments for the od-regenerative cycles of both the GSMP and its underlying general state space Markov chain.

1 Introduction

A wide variety of manufacturing, computer, transportation, telecommunication, and work-flow systems can usefully be viewed as *discrete-event stochastic systems*. Such systems evolve over continuous time and make stochastic state transitions when events associated with the occupied state occur; the state transitions occur only at an increasing sequence of random times. The underlying stochastic process of a discrete-event system records the state as it evolves over continuous time and has piecewise-constant sample paths.

The usual model for the underlying process of a complex discrete-event stochastic system is the *generalized semi-Markov process* (GSMP); see, for example, [5, 15, 16, 19, 20, 23]. In a GSMP, events associated with a state compete to trigger the next state transition and each set of trigger events has its own probability distribution for determining the new state. At each state transition, new events may be scheduled. For each of these new events, a clock indicating the time until

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the event is scheduled to occur is set according to a probability distribution that depends on the current state, the new state, and the set of events that trigger the state transition. These clocks, along with the speeds at which the clocks run down, determine when the next state transition occurs and which of the scheduled events actually trigger this state transition. A GSMP is formally defined in terms of a general state space Markov chain $\{ (S_n, C_n) : n \geq 0 \}$ that records the state of the system, together with the clock readings, at successive state transitions. The GSMP model either subsumes or is closely related to a number of important applied probability models such as continuous time Markov chains, semi-Markov processes, Markovian and non-Markovian multiclass networks of queues [20], and stochastic Petri nets [14].

Strong laws of large numbers (SLLNs) and central limit theorems (CLTs) formalize the notion of "stability" for the underlying stochastic process $\{ X(t) : t \geq 0 \}$ of a GSMP. These limit theorems also provide approximations for cumulative-reward distributions, confidence intervals for statistical estimators, and efficiency criteria for simulation algorithms. In more detail, an SLLN asserts the existence of time-average limits of the form $r(f) = \lim_{t \to \infty} (1/t) \int_0^t f\big(X(u)\big)\, du$, where f is a real-valued function. If such an SLLN holds, then the quantity $\hat{r}(t) = (1/t) \int_0^t f\big(X(u)\big)\, du$ is a strongly consistent estimator for $r(f)$. Viewing $R(t) = \int_0^t f\big(X(u)\big)\, du$ as the cumulative "reward" earned by the system in the interval $[0, t]$, the SLLN also asserts that $R(t)$ can be approximated by the quantity $r(f)t$ when t is large. Central limit theorems (CLTs) serve to illuminate the rate of convergence in the SLLN, to quantify the precision of $\hat{r}(t)$ as an estimator of $r(f)$, and to provide approximations for the distribution of the cumulative reward $R(t)$ at large values of t. The ordinary form of the CLT asserts that under appropriate regularity conditions, the quantity $\hat{r}(f)$—suitably normalized—converges in distribution to a standard normal random variable. An ordinary CLT often can be strengthened to a *functional central limit theorem* (FCLT); see, for example, [3, 4]. Roughly speaking, a stochastic process with time-average limit r obeys an FCLT if the associated cumulative (i.e., time-integrated) process—centered about the deterministic function $g(t) = rt$ and suitably compressed in space and time—converges in distribution to a standard Brownian motion as the degree of compression increases. A variety of estimation methods such as the method of batch means (with a fixed number of batches) are known to yield asymptotically valid confidence intervals for $r(f)$ when an FCLT holds [8]. Moreover, FCLT's can be used to analyze the behavior of the reward process $\{ R(t) : t \geq 0 \}$ over finite time intervals. For example, it is possible to identify a deterministic affine function g such that, with probability approximately equal to a specified value, we have $R(u) \leq g(u)$ for $0 \leq u \leq t$. Also of interest are "discrete time" SLLNs and FCLTs for the processes of the form $\{ \tilde{f}(S_n, C_n) : n \geq 0 \}$.

Given the central role played by the GSMP model in both theory and applications, it is fundamentally important to obtain the "right" conditions underlying basic limit theorems such as the SLLN and FCLT. Limit theory for semi-Markov processes [7] shows that the right general conditions involve some form of structural irreducibility, as well as finite first (resp., second) moments on the holding-time distribution in the case of the SLLN (resp., of the FCLT). Such conditions are "minimal" in that if we allow them to be violated, then we can easily find models for which the conclusion of the SLLN or FCLT fails to hold. In this paper, we provide new SLLNs and FCLTs for finite-state GSMPs under an irreducibility assumption and moment conditions that are comparable to the minimal conditions for semi-Markov processes. The moment conditions are substantially weaker than those given in [13]. Unlike with semi-Markov processes, we impose a positive-density condition on the clock-setting distributions. Some such condition is needed in the face of the additional complexity caused by the presence of multiple clocks; we show that in the absence of such a condition the SLLN and FCLT can fail to hold. We obtain our limit theorems by using Lyapunov-function arguments to show that the underlying chain of a GSMP is an od-regenerative process with finite cycle moments. The desired results then follow from limit theorems for od-regenerative processes.

2 Generalized Semi-Markov Processes

We briefly review the notation for, and definition of, a GSMP. Following [20], let $E = \{e_1, e_2, \ldots, e_M\}$ be a finite set of *events* and S be a finite or countably infinite set of *states*. For $s \in S$, let $s \mapsto E(s)$ be a mapping from S to the nonempty subsets of E; here $E(s)$ denotes the set of all events that can occur when the process is in state s. An event $e \in E(s)$ is said to be *active* in state s. When the process is in state s, the occurrence of one or more active events triggers a state transition. Denote by $p(s'; s, E^*)$ the probability that the new state is s' given that the events in the set E^* ($\subseteq E(s)$) occur simultaneously in state s. A "clock" is associated with each event. The clock reading for an active event indicates the remaining time until the event is scheduled to occur. These clocks, along with the speeds at which the clocks run down, determine which of the active events actually trigger the next state transition. Denote by $r(s, e)$ (≥ 0) the *speed* (finite, deterministic rate) at which the clock associated with event e runs down when the state is s; we assume that, for each $s \in S$, we have $r(s, e) > 0$ for some $e \in E(s)$. Typically in applications, all speeds for active events are equal to 1; zero speeds can be used to model preemptive-resume behavior. Let $C(s)$ be the set of possible *clock-reading vectors* when the state is s:

$$C(s) = \left\{ c = (c_1, \ldots, c_M) \colon c_i \in [0, \infty) \text{ and } c_i > 0 \text{ if and only if } e_i \in E(s) \right\}.$$

The ith component of a clock-reading vector $c = (c_1, \ldots, c_M)$ is the clock reading associated with event e_i.) Beginning in state s with clock-reading vector $c = (c_1, \ldots, c_M) \in C(s)$, the time $t^*(s, c)$ to the next state transition is given by

$$t^*(s, c) = \min_{\{\, i \colon e_i \in E(s) \,\}} c_i / r(s, e_i), \tag{2.1}$$

where $c_i / r(s, e_i)$ is taken to be $+\infty$ when $r(s, e_i) = 0$. The set of events $E^*(s, c)$ that trigger the next state transition is given by

$$E^*(s, c) = \{\, e_i \in E(s) \colon c_i - t^*(s, c) r(s, e_i) = 0 \,\}.$$

At a transition from state s to state s' triggered by the simultaneous occurrence of the events in the set E^*, a finite clock reading is generated for each *new event* $e' \in N(s'; s, E^*) = E(s') - (E(s) - E^*)$. Denote the *clock-setting distribution function* (that is, the distribution function of such a new clock reading) by $F(\,\cdot\,; s', e', s, E^*)$. We assume that $F(0; s', e', s, E^*) = 0$, so that new clock readings are a.s. positive, and that $\lim_{x\to\infty} F(x; s', e', s, E^*) = 1$, so that each new clock reading is a.s. finite. For each *old event* $e' \in O(s'; s, E^*) = E(s') \cap \big(E(s) - E^*\big)$, the old clock reading is kept after the state transition. For $e' \in (E(s) - E^*) - E(s')$, event e' is cancelled and the clock reading is discarded. When E^* is a singleton set of the form $E^* = \{\, e^* \,\}$, we write $p(s'; s, e^*) = p(s'; s, \{\, e^* \,\})$, $O(s'; s, e^*) = O(s'; s, \{\, e^* \,\})$, and so forth. The GSMP is a continuous-time stochastic process $\{\, X(t) \colon t \geq 0 \,\}$ that records the state of the system at time t.

Formal definition of the process $\{\, X(t) \colon t \geq 0 \,\}$ is in terms of a general state space Markov chain $\{\, (S_n, C_n) \colon n \geq 0 \,\}$ that describes the process at successive state-transition times. Heuristically, S_n represents the state and $C_n = (C_{n,1}, \ldots, C_{n,M})$ represents the clock-reading vector just after the nth state transition; see [20] for a formal definition of the chain. The chain takes values in the set $\Sigma = \bigcup_{s\in S} \big(\{\, s \,\} \times C(s)\big)$. Denote by μ the *initial distribution* of the chain; for a subset $B \subseteq \Sigma$, the quantity $\mu(B)$ represents the probability that $(S_0, C_0) \in B$. We use the notations P_μ and E_μ to denote probabilities and expected values associated with the chain, the idea being to emphasize the dependence on the initial distribution μ; when the initial state of the underlying chain is equal to some $(s, c) \in \Sigma$ with probability 1, we write $P_{(s,c)}$ and $E_{(s,c)}$. The symbol P^n denotes the n-step *transition kernel* of the chain: $P^n\big((s, c), A\big) = P_{(s,c)} \{\, (S_n, C_n) \in A \,\}$ for $(s, c) \in \Sigma$ and $A \subseteq \Sigma$; when $n = 1$ we simply write P to denote the 1-step transition kernel.

We construct a continuous time process $\{X(t) \colon t \geq 0\}$ from the chain $\{\, (S_n, C_n) \colon n \geq 0 \,\}$ in the following manner. Let ζ_n $(n \geq 0)$ be the (nonnegative, real-valued) time of the nth state transition: $\zeta_0 = 0$ and

$$\zeta_n = \sum_{j=0}^{n-1} t^*(S_j, C_j)$$

for $n \geq 1$. Let $\Delta \notin S$ and set

$$X(t) = \begin{cases} S_{N(t)} & \text{if } N(t) < \infty; \\ \Delta & \text{if } N(t) = \infty, \end{cases} \tag{2.2}$$

where

$$N(t) = \sup\{\, n \geq 0 \colon \zeta_n \leq t \,\}\,.$$

The stochastic process $\{\, X(t) \colon t \geq 0 \,\}$ defined by (2.2) is the GSMP. By construction, the GSMP takes values in the set $S \cup \{\, \Delta \,\}$ and has piecewise constant, right-continuous sample paths. We assume throughout that

$$P_\mu\Big\{ \sup_{n\geq 0} \zeta_n = \infty \Big\} = 1 \tag{2.3}$$

so that only a finite number of state transitions occur in any finite time interval. An argument as in Theorem 3.13 of [14, Ch. 3] shows that this condition holds, for example, whenever S is finite.

3 SLLNs and FCLTs for Generalized Semi-Markov Processes

3.1 Preliminaries

To prepare for the various limit theorems below, we introduce some notation and terminology. For a GSMP with state space S and event set E and for $s, s' \in S$ and $e \in E$, write $s \xrightarrow{e} s'$ if $p(s'; s, e)r(s, e) > 0$ and write $s \to s'$ if $s \xrightarrow{e} s'$ for some $e \in E(s)$. Also write $s \rightsquigarrow s'$ if either $s \to s'$ or there exist states $s_1, s_2, \ldots, s_n \in S$ ($n \geq 1$) such that $s \to s_1 \to \cdots \to s_n \to s'$.

Definition 3.1 A GSMP is *irreducible* if $s \rightsquigarrow s'$ for each $s, s' \in S$.

Recall that a nonnegative function G is a *component* of a distribution function F if G is not identically equal to 0 and $G \leq F$. If G is a component of F and G is absolutely continuous, so that G has a density function g, then we say that g is a *density component* of F.

Assumption PD(q), defined below, encapsulates the key conditions that we impose on the building blocks of a GSMP to obtain limit theorems.

Definition 3.2 *Assumption PD*(q) holds for a specified GSMP and real number $q \geq 0$ if

(i) the state space S of the GSMP is finite;

(ii) the GSMP is irreducible;

(iii) all speeds of the GSMP are positive; and

(iv) there exists $\bar{x} \in (0, \infty)$ such that each clock-setting distribution function $F(\,\cdot\,; s', e', s, E^*)$ of the GSMP has finite qth moment and a density component that is positive and continuous on $(0, \bar{x})$.

Observe that when Assumption PD(q) holds for some $q \geq 0$, the "infinite lifetime" condition in (2.3) holds and, with probability 1, events never occur simultaneously. Moreover, Assumption PD(r) holds for $0 \leq r < q$.

Recall that a probability distribution π is *invariant* with respect to a Markov chain $\{\, Z_n \colon n \geq 0 \,\}$ with transition kernel P and (possibly uncountable) state space Γ if and only if $\int P(z, A)\, \pi(dz) = \pi(A)$ for each measurable set $A \subseteq \Gamma$. The following result is a consequence of Proposition 3.13 in [13] and Theorem 4.5 in [13].

Proposition 3.3 *Suppose that Assumption PD*(1) *holds for a* GSMP. *Then there exists a unique invariant distribution* π *for the underlying chain* $\{\, (S_n, C_n) \colon n \geq 0 \,\}$.

Given an invariant distribution π for the underlying chain together with a real-valued function $\tilde{f}$ defined on Σ, we often write $\pi(\tilde{f}) = E_\pi\, [\tilde{f}(S_0, C_0)]$.

In the following sections, we denote by $C[0,1]$ the space of continuous real-valued functions on $[0,1]$ and by $\Rightarrow$ weak convergence on $C[0,1]$; see [3, 4]. Weak convergence on $C[0,1]$ generalizes to a sequence of random functions—i.e., a sequence of stochastic processes—the usual notion of convergence in distribution of a sequence of random variables.

3.2 Main Results: Discrete Time

Consider a GSMP $\{ X(t)\colon t \geq 0 \}$ with state space S and underlying chain $\{ (S_n, C_n)\colon n \geq 0 \}$ having state space Σ. Recall the definition of the holding-time function t^* in (2.1) and denote by $\mathcal{G}$ the set of real-valued functions defined on Σ. For $u \geq 0$, set

$$\mathcal{H}_u = \Big\{ h \in \mathcal{G}\colon |h(s,c)| \leq a + b\,\big(t^*(s,c)\big)^u \text{ for some } a, b \geq 0 \text{ and all } (s,c) \in \Sigma \Big\}.$$

Also write $x \vee y = \max(x,y)$ and, for a real-valued function g, write $|g|$ to denote the function defined by $|g|(x) = |g(x)|$. We now state a discrete-time SLLN and FCLT for the underlying chain of a GSMP; see Section 4 for proofs.

Theorem 3.4 *Suppose that Assumption PD($u \vee 1$) holds for some $u \geq 0$, so that there exists a unique invariant distribution π for the underlying chain $\{ (S_n, C_n)\colon n \geq 0 \}$. Then $\pi(|\tilde{f}|) < \infty$ and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}(S_n, C_n) = \pi(\tilde{f}) \ \text{a.s.}$$

for any $\tilde{f} \in \mathcal{H}_u$.

Observe that the time-average limit does not depend on the initial distribution μ.

Given a GSMP satisfying Assumption PD(1)—so that there exists a unique invariant distribution π for the underlying chain $\{ (S_n, C_n)\colon n \geq 0 \}$—along with a measurable function $\tilde{f}\colon \Sigma \mapsto \Re$ such that $\pi(|\tilde{f}|) < \infty$, define a sequence of $C[0,1]$-valued random functions $U_1(\tilde{f}), U_2(\tilde{f}), \ldots$ by setting

$$U_n(\tilde{f})(t) = \frac{1}{\sqrt{n}} \int_0^{nt} \Big(\tilde{f}(S_{\lfloor u \rfloor}, C_{\lfloor u \rfloor}) - \pi(\tilde{f}) \Big)\, du$$

for $0 \leq t \leq 1$ and $n \geq 0$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Denote by $W = \{ W(t)\colon 0 \leq t \leq 1 \}$ a standard Brownian motion on $[0,1]$; see, e.g., [3].

Theorem 3.5 *Let $u \geq 0$ and $\tilde{f} \in \mathcal{H}_u$. If Assumption PD$\big(2(u \vee 1)\big)$ holds, then there exists a finite constant $\tilde{\sigma}(\tilde{f}) \geq 0$ such that $U_n(\tilde{f}) \Rightarrow \tilde{\sigma}(\tilde{f}) W$ as $n \to \infty$.*

A variant of the foregoing result asserts weak convergence to a limiting Brownian motion on $D[0,1]$, the space of real-valued functions on $[0,1]$ that are right-continuous and have limits from the left. The statement of this theorem is identical to that of Theorem 3.5, except that the sequence $U_1(\tilde{f}), U_2(\tilde{f}), \ldots$ is defined by setting

$$U_n(\tilde{f})(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor} \big(\tilde{f}(S_j, C_j) - \pi(\tilde{f}) \big)$$

for $0 \leq t \leq 1$ and $n \geq 0$. The proof is essentially identical to that of Theorem 3.5.

3.3 Main Results: Continuous Time

We now give limit theorems in continuous time; proofs are provided in Section 4. Given an invariant distribution π for the underlying chain of a GSMP together with a function $f\colon S \mapsto \Re$, set

$$r(f) = \frac{\pi(ft^*)}{\pi(t^*)},$$

where t^* is the holding time function and $(ft^*)(s,c) = f(s)t^*(s,c)$ for $(s,c) \in \Sigma$.

Theorem 3.6 *Suppose that Assumption PD*(1) *holds. Then* $r(|f|) < \infty$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(X(u)\big)\,du = r(f) \;\; a.s.$$

for any real-valued function f *defined on* S.

As with the previous SLLN, the time-average limit does not depend on the initial distribution μ.

Given a GSMP satisfying Assumption PD(1) along with a real-valued function f defined on S such that $r(|f|) < \infty$, set

$$U_\nu(f)(t) = \frac{1}{\sqrt{\nu}} \int_0^{\nu t} \Big(f\big(X(u)\big) - r(f)\Big)\,du$$

for $0 \le t \le 1$ and $\nu \in \Re_+$; each random function $U_\nu(f)$ is an element of $C[0,1]$.

Theorem 3.7 *Suppose that Assumption PD*(2) *holds, and let* f *be a real-valued function defined on* S*. Then there exists a finite constant* $\sigma(f) \ge 0$ *such that* $U_\nu(f) \Rightarrow \sigma(f)W$ *as* $\nu \to \infty$.

3.4 Discussion

The conclusions of Theorems 3.4 and 3.5 hold for a function $f \in \mathcal{H}_u$ $(u \ge 1)$ under the respective assumptions PD(u) and PD($2u$), and the conclusions of Theorems 3.6 and 3.7 hold under the respective assumptions PD(1) and PD(2). Versions of the foregoing theorems are proved in [13] under substantially stronger moment conditions. Specifically, the SLLN and FCLT for the underlying chain hold for a function $f \in \mathcal{H}_u$ under the respective assumptions PD($u+1$) and PD($2u+3$), and the corresponding limit theorems for the process $\{\, X(t)\colon t \ge 0 \,\}$ hold under the respective assumptions PD(2) and PD(5).

The moment conditions in Theorems 3.6 and 3.7 are natural in light of known conditions [7] for semi-Markov processes and continuous-time Markov chains. The appropriateness of the moment conditions in Theorems 3.4 and 3.5 may not be quite as apparent. For example, it may not be clear why Theorem 3.5 requires finite second moments on the clock-setting distributions even when $\tilde{f}(s,c) \equiv g(s)$ for some function g, so that the constant u in the theorem can be taken as 0. The following example shows that the conclusion of Theorem 3.5 can fail when clock-setting distribution functions are allowed to have infinite second moments.

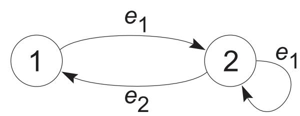

Figure 1: State transition diagram for GSMP of Example 3.8.

Example 3.8 Consider a GSMP with unit speeds, state space $S = \{1, 2\}$, event set $E = \{e_1, e_2\}$ and active event sets given by $E(1) = \{e_1\}$ and $E(2) = \{e_1, e_2\}$. The state-transition probabilities are

$$p(2; 1, e_1) = p(2; 2, e_1) = p(1; 2, e_2) = 1$$

(see Figure 1). The clock-setting distribution functions have the simple form $F(\cdot\,; e_i)$ for $i = 1, 2$. Denote by α_i and β_i the first and second moment of $F(\cdot\,; e_i)$. We assume that $\alpha_1, \alpha_2, \beta_1 < \infty$ and $\beta_2 = \infty$. We also assume that $F(\cdot\,; e_1)$ and $F(\cdot\,; e_2)$ each have a density function that is positive on $[0, \infty)$.

Set $\theta(-1) = -1$ and $\theta(n) = \inf\{k > \theta(n-1) \colon S_{\theta(n)} = 1\}$ for $n \geq 0$. Because only one clock is active in state 1, the underlying chain probabilistically restarts whenever it hits the set $\{1\} \times C(1)$. Because Assumption PD(1) holds, Proposition 4.8 below implies that the random indexes $\{\theta(n) \colon n \geq 0\}$ form a sequence of classical "regeneration" points for the underlying chain—see Section 4.1—and that the cycle length $\eta_1 = \theta(1) - \theta(0)$ has finite mean. It follows from Glynn and Whitt [10] that a necessary condition for the conclusion of Theorem 3.5 to hold with $\tilde{f}(s, c) = s$ is that η_1 have finite second moment. Observe that η_1 is distributed as $N(T) + 1$, where $\{N(t) \colon t \geq 0\}$ is a renewal counting process with inter-renewal distribution function $F(\cdot\,; e_1)$ and T is an independent sample from $F(\cdot\,; e_2)$. Using the Cauchy-Schwartz inequality together with a standard result for renewal counting processes [1, p. 158], we have $E[N^2(t)] \geq E^2[N(t)] \geq t^2/\alpha_1^2$ for $t \geq 0$. Thus

$$E[\eta_1^2] \geq E[N^2(T)] = E\big[E[N^2(T) \mid T]\big] \geq E[T^2/\alpha_1^2] = \infty,$$

so that the conclusion of Theorem 3.5 fails to hold. □

A slight modification of the foregoing example shows that conclusion of Theorem 3.4 can fail to hold if we allow clock-setting distributions to have infinite mean.

Our assumption of positive density components for the clock-setting distributions is by no means necessary—it is easy to construct GSMPs that violate this assumption but still satisfy SLLNs and FCLTs. The following example, however, shows that some sort of assumption is needed in order to ensure that value of a time-average limit does not depend upon the initial distribution.

Example 3.9 (An irreducible GSMP with no unique time-average limit) Consider a GSMP with unit speeds, state space $S = \{1, 2, 3, 4\}$, event set $E = \{e_1, e_2\}$ and active event sets given by $E(1) = E(3) = \{e_1, e_2\}$ and $E(2) = E(4) = \{e_2\}$. The state-transition probabilities are

$$p(1; 3, e_1) = p(3; 1, e_1) = 1$$

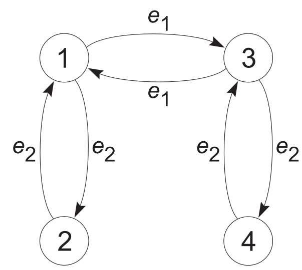

Figure 2: State transition diagram for GSMP of Example 3.9.

and

$$p(1;2,e_2) = p(2;1,e_2) = p(3;4,e_2) = p(4;3,e_2) = 1$$

(see Figure 2). Observe that this GSMP is irreducible in the sense of Definition 3.1. Suppose that each successive new clock reading for event e_i $(i = 1,2)$ is uniformly distributed on a specified interval $[a_i, b_i]$, and that $0 \leq a_2 < b_2 < a_1 < b_1$. Then with probability 1 event e_2 always occurs before event e_1 whenever both events simultaneously become active. It follows that if the initial state is equal to 1 or 2, then the GSMP never hits state 3 or 4; if the initial state is equal to 3 or 4, then the GSMP never hits state 1 or 2. Thus, in general, the value of a limit of the form $\lim_{t\to\infty}(1/t)\int_0^t f\bigl(X(u)\bigr)\,du$ depends on the initial distribution. Similar observations hold for the underlying chain. Of course, this GSMP does not satisfy Assumption PD(q) for any $q \geq 0$ since the clock-setting distribution function for transition e_1 does not have a density component that is positive on an interval of the form $(0, \bar{x})$. □

In the continuous-time setting, the results in this paper focus on rewards that accrue continuously at rate $f(s)$ whenever the GSMP is in state $s \in S$. It is not difficult to extend our results to handle "impulse rewards," e.g., a reward of the form $g(s';s,E^*)$ that accrues whenever the simultaneous occurrence of the events in E^* triggers a transition from s to s'. The idea is to consider the Markov chain $\{\,(S_n, C_n, S_{n+1}, C_{n+1})\colon n \geq 0\,\}$, which inherits the stability properties of the underlying chain.

4 Proofs

In this section we establish the results in Sections 3.2 and 3.3. To this end, we first review some limit theory for od-regenerative processes and then recall some conditions under which a GSMP both satisfies a drift criterion for recurrence and consequently enjoys od-regenerative structure.

4.1 Limit Theorems for OD-Regenerative Processes

Thorough discussions of od-regenerative and related processes can be found, for example, in [1, 6, 14, 21, 22]. We focus on processes that evolve over continuous time; to obtain corresponding results for a discrete-time process $\{ X_n \colon n \geq 0 \}$, apply the continuous-time theory to the process $\{ X_{\lfloor t \rfloor} \colon t \geq 0 \}$.

For the sequence of random times $\{ T_k \colon k \geq 0 \}$ defined below, set $\tau_k = T_k - T_{k-1}$ for $k \geq 1$.

Definition 4.1 The stochastic process $\{ X(t) \colon t \geq 0 \}$ with state space S is an *od-regenerative process* in continuous time if there exists an increasing sequence $0 \leq T_0 < T_1 < T_2 < \cdots$ of a.s. finite random times such that, for $k \geq 1$, the post-T_k process $\{ X(T_k + t) \colon t \geq 0;\ \tau_{k+l} \colon l \geq 1 \}$

(i) is distributed as the post-T_0 process $\{ X(T_0 + t) \colon t \geq 0;\ \tau_l \colon l \geq 1 \}$; and

(ii) is independent of the pre-T_{k-1} process $\{ X(t) \colon 0 \leq t < T_{k-1};\ \tau_1, \ldots, \tau_{k-1} \}$.

The od-regeneration points serve to decompose sample paths of $\{ X(t) \colon t \geq 0 \}$ into one-dependent stationary *cycles*. The random variable τ_k defined above is the length of the kth cycle. A classical regenerative process is a special case of an od-regenerative process in which the cycles are i.i.d..

When $T_0 = 0$ the process $\{ X(t) \colon t \geq 0 \}$ is *nondelayed*; otherwise, it is called *delayed*. For a delayed $\{ X(t) \colon t \geq 0 \}$, the "0th cycle" $\{ X(t) \colon 0 \leq t < T_0 \}$ need not have the same distribution as the other cycles. Similarly, the length of this cycle—denoted by τ_0—need not have the same distribution as τ_1, τ_2, and so forth.

We first state an SLLN for od-regenerative processes. Given such a process $\{ X(t) \colon t \geq 0 \}$ with state space S and od-regeneration points $\{ T_k \colon k \geq 0 \}$, along with a real-valued function f defined on S, set

$$Y_k(f) = \int_{T_{k-1}}^{T_k} f\bigl(X(u)\bigr)\, du$$

for $k \geq 0$. (Take $T_{-1} = 0$.) It follows from the definition of an od-regenerative process that the sequence $\{ \bigl(Y_k(f), \tau_k\bigr) \colon k \geq 1 \}$ consists of identically distributed random pairs. Set

$$r(f) = \frac{E\,[Y_1(f)]}{E\,[\tau_1]}$$

and observe that $r(f)$ is well defined and finite if and only if $r(|f|) < \infty$.

Proposition 4.2 *Suppose that $E\,[\tau_1] < \infty$. Then $r(|f|) < \infty$ and*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f\bigl(X(u)\bigr)\, du = r(f) \ \text{a.s.}$$

for any real-valued function f such that $Y_0(|f|) < \infty$ a.s. and $E\,[Y_1(|f|)] < \infty$.

The proof of this result is almost identical to the proof of the SLLN for classical regenerative processes [1]. The only difference is that the proof rests on the SLLN for m-dependent random variables [2, p. 86] rather than the classical SLLN for i.i.d. random variables.

We now state an FCLT for wide sense regenerative processes. Given a wide sense regenerative process $\{ X(t) \colon t \geq 0 \}$ with state space S and a real-valued function f for which $r(|f|) < \infty$, set

$$U_\nu(f)(t) = \frac{1}{\sqrt{\nu}} \int_0^{\nu t} \Bigl(f\bigl(X(u)\bigr) - r(f) \Bigr)\, du$$

for $0 \le t \le 1$ and $\nu \in \Re_+$. Also set

$$\sigma^2(f) = \frac{\mathrm{Var}_\mu\left[Y_1(f) - r(f)\tau_1\right] + 2\mathrm{Cov}_\mu\left[Y_1(f) - r(f)\tau_1, Y_2(f) - r(f)\tau_2\right]}{E_\mu\left[\tau_1\right]}.$$

Proposition 4.3 *Let $\{\, X(t)\colon t \ge 0 \,\}$ be an od-regenerative process with state space S and let f be a real-valued function defined on S. Suppose that $Y_0(|f|) < \infty$ a.s. and $E_\mu\left[Y_1^2(|f|) + \tau_1^2\right] < \infty$. Then $U_\nu(f) \Rightarrow \sigma(f)W$ as $\nu \to \infty$.*

The proof of Proposition 4.3 rests on the FCLT for "mixing" stationary random variables [3, Th. 19.2] together with a "random time change" result [3, Sec. 14]; see [13] for further details.

4.2 Harris Recurrence and OD-Regenerative Structure in GSMPs

The key to our analysis of GSMPs is Proposition 4.6 below, which establishes conditions under which underlying chain satisfies a "drift" condition for stability. To prepare for this result, we review some terminology for a Markov chain $\{\, Z_n\colon n \ge 0 \,\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a (possibly uncountably infinite) state space Γ; see [17] for details. Such a chain is *ϕ-irreducible* if ϕ is a nontrivial measure on subsets of Γ and, for each $z \in \Gamma$ and subset $A \subseteq \Gamma$ with $\phi(A) > 0$, there exists $n > 0$—possibly depending on both z and A—such that $P^n(z, A) > 0$. [Here P^n is the n-step transition kernel for the chain.] A ϕ-irreducible chain is *Harris recurrent* if $P_z\{\, Z_n \in A \text{ i.o.} \,\} = 1$ for all $z \in \Gamma$ and $A \subseteq \Gamma$ with $\phi(A) > 0$. A Harris recurrent chain admits an invariant distribution π_0 that is unique up to constant multiples. If $\pi_0(\Gamma) < \infty$, then $\pi(\,\cdot\,) = \pi_0(\,\cdot\,)/\pi(\Gamma)$ is an invariant probability distribution for the chain. A Harris recurrent chain that admits an invariant probability distribution is called *positive Harris recurrent*. A subset $B \subseteq \Gamma$ is *petite* with respect to the chain if there exists a probability distribution q on the nonnegative integers and a nontrivial measure ψ on subsets of Γ such that

$$\inf_{z \in B} \sum_{n=0}^{\infty} q(n) P^n\big(z, A\big) \ge \psi(A)$$

for $A \subseteq \Gamma$.

Now consider a GSMP with event set $E = \{\, e_1, \ldots, e_M \,\}$ and underlying chain $\{\, (S_n, C_n)\colon n \ge 0 \,\}$ taking values in Σ. Set

$$H_b = \big(S \times [0, b]^M\big) \cap \Sigma \tag{4.4}$$

for $b > 0$ and define the "stochastic Lyapunov function"

$$h_q(s, c) = 1 + \max_{1 \le i \le M} c_i^q$$

for $s \in S$, $c = (c_1, c_2, \ldots, c_M) \in C(s)$, and $q \ge 0$. Whenever Assumption PD holds, we define $\bar\phi$ to be the unique measure on subsets of Σ such that

$$\bar\phi\big(\{\, s \,\} \times [0, x_1] \times [0, x_2] \times \cdots \times [0, x_M]\big) = \prod_{\{i\colon e_i \in E(s)\}} \min(x_i, \bar x) \tag{4.5}$$

for all $s \in S$ and $x_1, x_2, \ldots, x_M \ge 0$. If, for example, a set $B \subseteq \Sigma$ is of the form $B = \{\, s \,\} \times A$ with $E(s) = E$, then $\bar\phi(B)$ is equal to the Lebesgue measure of the set $A \cap [0, \bar x]^M$.

Proposition 4.6 *If Assumption PD*(0) *holds, then*

(i) the underlying chain $\{\, (S_n, C_n)\colon n \geq 0 \,\}$ *is* $\bar\phi$*-irreducible, where* $\bar\phi$ *is defined by* (4.5)*, and*

(ii) for each $b > 0$ *the set* H_b *defined by* (4.4) *is petite with respect to* $\{\, (S_n, C_n)\colon n \geq 0 \,\}$.

If, moreover, Assumption PD(q) *holds for some* $q \geq 1$*, then for all sufficiently large* b

(iii) $\sup_{(s,c)\in H_b} E_{(s,c)}\left[h_q(S_M, C_M) - h_q(S_0, C_0)\right] < \infty$*, and*

(iv) there exists $\beta \in (0,1)$ *such that*

$$E_{(s,c)}\left[h_q(S_M, C_M) - h_q(S_0, C_0)\right] \leq -\beta h_q(s,c) \tag{4.7}$$

for $(s,c) \in \Sigma - H_b$.

See [13] for a proof of this result. Combining Proposition 4.6 with [13, Prop. 3.13], we obtain the following recurrence result.

Proposition 4.8 *If Assumption PD*(1) *holds, then* $\{\, (S_n, C_n)\colon n \geq 0 \,\}$ *is positive Harris recurrent with recurrence measure* $\bar\phi$ *defined by* (4.5).

The foregoing propositions lead to a sufficient condition for od-regenerative structure in a GSMP.

Proposition 4.9 *Let* $\{\, (S_n, C_n)\colon n \geq 0 \,\}$ *be the underlying chain of a* GSMP*. If Assumption PD*(1) *holds, then there exists a sequence* $\{\, \theta(k)\colon k \geq 0 \,\}$ *of od-regeneration points for* $\{\, (S_n, C_n) : n \geq 0 \,\}$*. Moreover, the invariant distribution* π *of the chain has the representation*

$$\pi(A) = \frac{E_\mu\left[\sum_{n=\theta(0)}^{\theta(1)-1} 1_A(S_n, C_n)\right]}{E_\mu\left[\eta_1\right]}$$

for $A \subseteq \Sigma$*, where* $\eta_1 = \theta(1) - \theta(0)$ *and* 1_A *is the indicator function of the set* A.

The idea of the proof is as follows; see [13] for details. By Proposition 4.8, the underlying chain is Harris recurrent. By a well known result for Harris chains, there exists a set $\mathcal{C} \subseteq \Sigma$ with $\bar\phi(\mathcal{C}) > 0$ such that

$$P^r\big((s,c), \cdot\,\big) = \epsilon\lambda(\,\cdot\,) + (1-\epsilon)Q\big((s,c), \cdot\,\big), \qquad (s,c) \in \mathcal{C} \tag{4.10}$$

for some $r \geq 1$, $\epsilon \in (0,1]$, probability distribution λ, and transition kernel Q; see Asmussen [1, Sec. VI.3], Glynn and L'Ecuyer [9], and Meyn and Tweedie [17, Th. 5.2.3]. Indeed, any subset $A \subseteq \Sigma$ with $\bar\phi(A) > 0$ contains such a $\mathcal{C}$-set; [17, Th. 5.2.2]. Observe that, since $\bar\phi(\mathcal{C}) > 0$, it follows that

$$P_\mu\{\, (S_n, C_n) \in \mathcal{C} \text{ i.o.} \,\} = 1.$$

The decomposition in (4.10) permits construction of a version of the underlying chain together with a sequence $\{\, \theta(k)\colon k \geq 0 \,\}$ of random indices that serve as od-regeneration points. The construction uses a sequence $\{\, I_n\colon n \geq 0 \,\}$ of i.i.d. Bernoulli random variables with $P_\mu\{\, I_n = 1 \,\} =$

$1 - P_\mu\{ I_n = 0 \} = \epsilon$. The idea is to generate successive states of the chain according to the initial distribution μ and one-step transition kernel P until the first time $M \geq 0$ such that $(S_M, C_M) \in \mathcal{C}$. If $I_M = 1$, then generate (S_{M+r}, C_{M+r}) according to λ; if $I_M = 0$, then generate (S_{M+r}, C_{M+r}) according to $Q\big((S_M, C_M), \cdot\big)$. Next, generate the intermediate states $\{ (S_n, C_n) \colon M+1 \leq n < M+r \}$ according to an appropriate conditional distribution (conditioned on the endpoint values (S_M, C_M) and (S_{M+r}, C_{M+r})). Now iterate this procedure starting from state (S_{M+r}, C_{M+r}). The successive times $\theta(0), \theta(1), \ldots$ at which the state of the chain is distributed according to λ form a sequence of od-regeneration points. [Observe that the length of each cycle is greater than or equal to r. In general, the conditioning on (S_M, C_M) and (S_{M+r}, C_{M+r}) mentioned above results in statistical dependence between $(S_{\theta(n)}, C_{\theta(n)})$ and $(S_{\theta(n)-r}, C_{\theta(n)-r})$ for each $n \geq 0$, which is why the cycles are one-dependent.] The second assertion of the proposition follows from Theorem VI.3.2 in [1].

4.3 Proof of the SLLNs and FCLTs

Under Assumption PD(1), Proposition 4.9 guarantees the existence of a sequence $\{ \theta(k) \colon k \geq 0 \}$ of od-regeneration points for the underlying chain $\{ (S_n, C_n) \colon n \geq 0 \}$ and a corresponding sequence $\big\{ \zeta_{\theta(k)} \colon k \geq 0 \big\}$ of od-regeneration points for the GSMP $\{ X(t) \colon t \geq 0 \}$. For a real-valued function $\tilde{f}$ defined on Σ, set

$$\tilde{Y}_i(\tilde{f}) = \sum_{j=\theta(i-1)}^{\theta(i)-1} \tilde{f}(S_n, C_n) \tag{4.11}$$

for $i \geq 0$. [Take $\theta(-1) = 0$.] Theorem 3.6 follows from Proposition 4.2 provided that the cycle length $\tau_1 = \zeta_{\theta(1)} - \zeta_{\theta(0)} = \tilde{Y}_1(t^*)$ has finite mean, and Theorem 3.7 follows from Proposition 4.3 provided that τ_1 has finite second moment. Similarly, Theorem 3.4 (resp., Theorem 3.5) follows from the discrete-time version of Proposition 4.2 (resp., Proposition 4.3) provided that the cycle length $\eta_1 = \theta(1) - \theta(0)$ and the cycle quantity $\tilde{Y}_1(|\tilde{f}|)$ have finite first (resp., second) moments. [In this connection, observe that $\tilde{Y}_0(|\tilde{f}|) < \infty$ a.s. because $\theta(0)$ is a.s. finite by Proposition 4.9 and each new clock reading is a.s. finite by definition.] To establish the desired limit theorems, it therefore suffices to prove the following general result on cycle moments.

Theorem 4.12 *Suppose that Assumption PD$\big(q(u \vee 1)\big)$ holds for some $q \in \{ 1, 2, \ldots \}$ and $u \geq 0$. Then $E_\mu[\tilde{Y}_1^q(|\tilde{f}|)] < \infty$ for any $\tilde{f} \in \mathcal{H}_u$, where $Y_1(|\tilde{f}|)$ is defined as in* (4.11).

We prove the assertion of Theorem 4.12 via a sequence of lemmas. Fix a compact set $B \subseteq \Sigma$ and denote by T_B the return time to B: $T_B = \inf\{ n > 0 \colon (S_n, C_n) \in B \}$. Lemma 4.15 below gives upper bounds on the moments of T_B. To prepare for this lemma, first observe that, by an argument that uses the drift condition (4.7) in Proposition 4.6 together with Dynkin's formula, we have

$$E_{(s,c)}\left[\sum_{n=0}^{T_B - 1} h_{q-1}(S_n, C_n) \right] \leq \gamma_q h_q(s, c) \tag{4.13}$$

for some positive constant $\gamma_q = \gamma_q(B) < \infty$ and all $(s,c) \in \Sigma$; see [17, Th. 14.2.3] for details. Next, fix finite positive constants $a_1 = 1, a_2, a_3, \ldots$ such that

$$n^{q+1} \le a_{q+1}(1^q + 2^q + \cdots + n^q) \tag{4.14}$$

for $n \ge 1$ and $q \in \{0,1,2,\ldots\}$—it is well known that such constants exist. Finally, set $b_q = \prod_{i=1}^{q}(a_i\gamma_i)$ for $q \ge 1$.

Lemma 4.15 *Suppose that Assumption PD(q) holds for some $q \in \{1,2,\ldots\}$. Then*

$$E_{(s,c)}[T_B^q] \le b_q h_q(s,c)$$

for $(s,c) \in \Sigma$.

Proof. Our proof is by induction on q. Fix $(s,c) \in \Sigma$ and observe that the desired result holds for $q = 1$ by virtue of (4.13). Assume for induction that the lemma holds for some $q \ge 1$ and observe that, by (4.14),

$$E_{(s,c)}[T_B^{q+1}] \le a_{q+1}E_{(s,c)}\left[\sum_{n=0}^{T_B-1}(T_B-n)^q\right] = a_{q+1}\sum_{n=0}^{\infty}E_{(s,c)}\left[(T_B-n)^q; T_B > n\right], \tag{4.16}$$

where the interchange of sum and expectation is justified by the nonnegativity of the summands. Using the Markov property together with the induction hypothesis, we find that

$$\begin{aligned} E_{(s,c)}[(T_B-n)^q; T_B > n] &= E_{(s,c)}\big[\, E_{(s,c)}[(T_B-n)^q; T_B > n \mid (S_k, C_k)\colon 0 \le k \le n]\,\big] \\ &= E_{(s,c)}\big[\, I(T_B > n)E_{(S_n,C_n)}[T_B^q]\,\big] \\ &\le E_{(s,c)}\big[\, I(T_B > n) b_q h_q(S_n, C_n)\,\big], \end{aligned} \tag{4.17}$$

where $I(A)$ is the indicator function for the event A. Substituting (4.17) into (4.16), interchanging sum and expectation, and applying (4.13), we find that

$$E_{(s,c)}[T_B^{q+1}] \le a_{q+1}b_q E_{(s,c)}\left[\sum_{n=0}^{T_B-1} h_q(S_n, C_n)\right] \le a_{q+1}\gamma_{q+1}b_q h_{q+1}(s,c) = b_{q+1}h_{q+1}(s,c),$$

and the desired result follows. □

The next step in the argument is to show that the discrete-time cycle length η_1 has finite qth moment under Assumption PD(q). To this end, we use the following fact: if $X_1, X_2, \ldots, X_k$ ($k \ge 1$) are nonnegative random variables and $a_1, a_2, \ldots, a_k$ are positive integers, then

$$E\left[X_1^{a_1}X_2^{a_2}\cdots X_k^{a_k}\right] \le E^{a_1/q}\left[X_1^q\right]E^{a_2/q}\left[X_2^q\right]\cdots E^{a_k/q}\left[X_k^q\right], \tag{4.18}$$

where $q = a_1 + a_2 + \cdots + a_k$. The inequality in (4.18) follows by an easy induction argument on k that uses Hölder's inequality.

Lemma 4.19 *Suppose that Assumption PD(q) holds for some $q \in \{1,2,\ldots\}$. Then $E_\mu[\eta_1^q] < \infty$.*

Proof. We give the proof under the simplifying assumption that (4.10) holds with $r = 1$; the extension to the general case is straightforward as in [18]. Let $\mathcal{C} \subset \Sigma$ be as in (4.10), and set

$$\alpha_q = \sup_{(s,c)\in\mathcal{C}} E_{(s,c)}[T_{\mathcal{C}}^q].$$

We can assume that $\mathcal{C}$ is compact, and it follows from Lemma 4.15 that $\alpha_q < \infty$. Assume for convenience that the initial state of the chain is an element of $\mathcal{C}$, and that the initial Bernoulli trial is successful (i.e., $I_0 = 1$), so that $\theta(0) = 1$. Denote by δ_i the number of state transitions between the $(i-1)$st and ith visit of the underlying chain to $\mathcal{C}$, where the 0th visit occurs at time 0. Also denote by N the number of returns to $\mathcal{C}$, up to and including the return that corresponds to the first successful Bernoulli trial after time 0. Observe that

$$E_\mu\left[\eta_1^q\right] = E_\mu\left[\left(\sum_{i=1}^N \delta_i\right)^q\right].$$

We can write

$$\left(\sum_{i=1}^N \delta_i\right)^q \le b_1 S_1 + b_2 S_2 + \cdots + b_m S_m,$$

where m and $b_1, b_2, \ldots, b_m$ are finite integers and each S_j is a sum of the form

$$S_j = \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_k=1}^N \delta_{i_1}^{a_1}\delta_{i_2}^{a_2}\cdots\delta_{i_k}^{a_k}.$$

Here the integers $k, a_1, a_2, \ldots, a_m$ are such that $k = k(j) \le q$, $a_l = a_l(j) \ge 1$ for $1 \le l \le k$, and $a_1 + \cdots + a_k = q$. It therefore suffices to show that each S_j has finite mean. Consider an arbitrary fixed value of j, and observe that, using (4.18),

$$\begin{aligned}
E_\mu\left[S_j\right] &= E_\mu\left[\sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_k=1}^N \delta_{i_1}^{a_1}\delta_{i_2}^{a_2}\cdots\delta_{i_k}^{a_k}\right] \\
&= \sum_{i_1=1}^\infty \sum_{i_2=1}^\infty \cdots \sum_{i_k=1}^\infty E_\mu\left[\delta_{i_1}^{a_1} I(N \ge i_1)\,\delta_{i_2}^{a_2} I(N \ge i_2)\cdots\delta_{i_k}^{a_k} I(N \ge i_k)\right] \qquad (4.20) \\
&\le \sum_{i_1=1}^\infty \sum_{i_2=1}^\infty \cdots \sum_{i_k=1}^\infty \left(\prod_{l=1}^k E_\mu^{a_l/q}\left[\delta_{i_l}^q I(N \ge i_l)\right]\right).
\end{aligned}$$

Let $\mathcal{F}_0 = \sigma\langle S_0, C_0, I_0\rangle$, that is, the σ-field generated by (S_0, C_0, I_0), and $\mathcal{F}_j = \sigma\langle (S_n, C_n, I_n) \colon 0 \le n \le \delta_1 + \cdots + \delta_j\rangle$ for $j \ge 1$. For each $i \ge 1$, observe that $I(N \ge i) \in \mathcal{F}_{i-1}$, so that

$$E_\mu[\delta_i^q I(N \ge i)] = E_\mu\big[I(N \ge i) E_\mu[\delta_i^q \mid \mathcal{F}_{i-1}]\big] \le \alpha_q P_\mu\left\{ N \ge i \right\}.$$

Using the foregoing inequality together with (4.20), we find that

$$E_\mu\left[S_j\right] \le \alpha_q \prod_{l=1}^k \left(\sum_{i=1}^\infty P_\mu^{a_l/q}\left\{ N \ge i \right\}\right) = \alpha_q \prod_{l=1}^k \left(\sum_{i=1}^\infty (1-\epsilon)^{a_l(i-1)/q}\right) < \infty$$

as desired. □

To complete the proof of Theorem 4.12, we need the following proposition.

Proposition 4.21 *Let $S_N = \sum_{n=1}^{N} X_n$, where $\{ X_n \colon n \geq 1 \}$ is a sequence of i.i.d. random variables and N is a stopping time with respect to an increasing sequence $\{ \mathcal{F}_n \colon n \geq 1 \}$ of σ-fields such that X_n is measurable with respect to $\mathcal{F}_n$ for $n \geq 1$ and independent of $\mathcal{F}_{n-1}$ for $n \geq 2$. Then for $r \geq 0$ there exists a finite constant b_r (depending only on r) such that*

$$E[|S_N|^r] \leq b_r E\left[|X_1|^r\right] E\left[N^r\right].$$

The proof of Proposition 4.21 is contained in the proof of Theorem I.5.2 in Gut [12].

Proof of Theorem 4.12. Fix q, u, and $\tilde{f} \in \mathcal{H}_u$. For ease of exposition, we assume that all speeds for active events are equal to 1 and that $F(\,\cdot\,; s', e', s, e^*) \equiv F(\,\cdot\,; e')$ for all s', e', s, and e^*. Denote by $A_{i,j}$ the value of the jth new clock reading generated for event e_i after time $\zeta_{\theta(0)}$, and by N_i the number of new clock readings generated for event e_i in the interval $(\zeta_{\theta(0)}, \zeta_{\theta(1)})$. Observe that

$$\tilde{Y}_1(|\tilde{f}|) \leq a\eta_1 + b\sum_{i=1}^{M} C^u_{\theta(0),i} + b\sum_{i=1}^{M}\sum_{j=1}^{N_i+r} A^u_{i,j}$$

for some $a, b \geq 0$, where r is as in (4.10). By Lemma 4.19, it therefore suffices to show that

$$E_\mu\left[\left(\sum_{j=1}^{N_i+r} A^u_{i,j}\right)^q\right] < \infty \tag{4.22}$$

and

$$E_\mu\left[C^{qu}_{\theta(0),i}\right] < \infty \tag{4.23}$$

for $1 \leq i \leq M$—this assertion follows from the elementary inequality

$$E\left[(X_1 + X_2 + \cdots + X_k)^q\right] \leq k^{q-1}\Big(E\left[X_1^q\right] + E\left[X_2^q\right] + \cdots + E\left[X_k^q\right]\Big),$$

which holds for any nonnegative random variables $X_1, X_2, \ldots, X_k$ ($k \geq 1$). To see that (4.22) holds, fix i and denote by $\kappa(i,j)$ the random index of the state transition at which $A_{i,j}$ is generated. Set $\mathcal{F}_j = \sigma\langle (S_n, C_n, I_n) \colon 0 \leq n \leq \kappa(i,j)\rangle$ for $j \geq 1$, and observe that (i) $A_{i,j} \in \mathcal{F}_j$ for $j \geq 1$, (ii) $A_{i,j}$ is independent of $\mathcal{F}_{j-1}$ for $j \geq 2$, and (iii) $N_i + r$ is a stopping time with respect to $\{ \mathcal{F}_n \colon n \geq 1 \}$. Moreover, since $N_i \leq \eta_1$, it follows from Lemma 4.19 that $E_\mu\left[(N_i + r)^q\right] < \infty$. An application of Proposition 4.21 now establishes (4.22). In light of (4.10), it can be seen that a sufficient condition for (4.23) to hold is

$$\sup_{(s,c)\in\mathcal{C}} \int_0^\infty x^{qu}\, dG_i(x; s, c) < \infty,$$

where $G_i(x; s, c) = P_{(s,c)}\{ C_{r,i} \leq x \}$. Observe that if X_i is distributed according to $G_i(x; s, c)$, then X_i is stochastically dominated by $W_i(c) = W_i(c_1, c_2, \ldots, c_M) = \max(c_i, B_{i,1}, B_{i,2}, \ldots, B_{i,r})$, where $B_{i,1}, B_{i,2}, \ldots, B_{i,r}$ are i.i.d. samples from $F(\,\cdot\,; e_i)$. Because $\mathcal{C}$ is assumed compact, there is a finite constant b such that $\max_{1\leq j\leq M} c_i \leq b$ for all $c = (c_1, c_2, \ldots, c_M)$ such that $(s, c) \in \mathcal{C}$. Thus

$$E\left[X_i^{qu}\right] \leq E\left[W_i^{qu}(c)\right] \leq b^{qu} + rE[B^{qu}_{i,1}] < \infty,$$

and the desired result follows. ∎

As an aside, it follows from the results in Section 4.1 that the limits $r(f)$ and $\pi(\tilde{f})$ in the SLLN's of Section 3 can be expressed as ratios of the form

$$r(f) = \frac{E_\mu\,[Y_1(f)]}{E_\mu\,[\tau_1]} \qquad \text{and} \qquad \pi(\tilde{f}) = \frac{E_\mu\,[\tilde{Y}_1(\tilde{f})]}{E_\mu\,[\eta_1]},$$

where $\tilde{Y}_1(\tilde{f}) = \sum_{j=\theta(0)}^{\theta(1)-1} \tilde{f}(S_n, C_n)$ and $Y_1(f) = \int_{\zeta_{\theta(0)}}^{\zeta_{\theta(1)}} f\big(X(u)\big)\,du$. Moreover, the variance constants $\sigma^2(f)$ and $\tilde{\sigma}^2(\tilde{f})$ can be expressed as

$$\sigma^2(f) = \frac{\mathrm{Var}_\mu\,[Z_1(f)] + 2\mathrm{Cov}_\mu\,[Z_1(f), Z_2(f)]}{E_\mu\,[\tau_1]}$$

and

$$\tilde{\sigma}^2(\tilde{f}) = \frac{\mathrm{Var}_\mu\,[\tilde{Z}_1(\tilde{f})] + 2\mathrm{Cov}_\mu\,[\tilde{Z}_1(\tilde{f}), \tilde{Z}_2(\tilde{f})]}{E_\mu\,[\eta_1]},$$

where $Z_k(f) = Y_k(f) - r(f)\tau_k$ and $\tilde{Z}_k(\tilde{f}) = \tilde{Y}_k(\tilde{f}) - \pi(\tilde{f})\eta_k$ for $k \geq 1$.

References

[1] Asmussen, S. (1987). *Applied Probability and Queues*. Wiley, New York.

[2] Billingsley, P. (1986). *Probability and Measure*. Wiley, New York.

[3] Billingsley, P. (1999). *Convergence of Probability Measures*. Wiley, New York, 2nd edn.

[4] Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.

[5] Glynn, P. W. (1989). A GSMP formalism for discrete event systems. *Proc. IEEE* **77**, 14–23.

[6] Glynn, P. W. (1994). Some topics in regenerative steady-state simulation. *Acta Appl. Math.* **34**, 225–236.

[7] Glynn, P. W. and Haas, P. J. (2003). A survey of functional central limit theorems for semi-Markov and related processes. Submitted for publication.

[8] Glynn, P. W. and Iglehart, D. L. (1990). Simulation output analysis using standardized time series. *Math. Oper. Res.* **15**, 1–16.

[9] Glynn, P. W. and L'Ecuyer, P. (1995). Likelihood ratio gradient estimation for stochastic recursions. *Adv. Appl. Probab.* **27**, 1019–1053.

[10] Glynn, P. W. and Whitt, W. (2002). Necessary conditions in limit theorems for cumulative processes. *Stochastic Process. Appl.* **98**, 199–209.

[11] Gnedenko, B. V. and Kovalenko, I. N. (1974). *Introduction to Queueing Theory*. Akademie-Verlag, Berlin, German edn.

[12] Gut, A. (1988). *Stopped Random Walks: Limit Theorems and Applications*. Springer-Verlag, New York.

[13] Haas, P. J. (1999). On simulation output analysis for generalized semi-Markov processes. *Comm. Statist. Stochastic Models* **15**, 53–80.

[14] Haas, P. J. (2002). *Stochastic Petri Nets: Modelling, Stability, Simulation*. Springer-Verlag, New York.

[15] Haas, P. J. and Shedler, G. S. (1987). Regenerative generalized semi-Markov processes. *Comm. Statist. Stochastic Models* **3**, 409–438.

[16] König, D., Matthes, K., and Nawrotzki, K. (1974). Unempfindlichkeitseigenshaften von Bedienungsprozessen. Appendix to [11].

[17] Meyn, S. P. and Tweedie, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer-Verlag, London.

[18] Rosenthal, J. S. (1995). Minorization conditions and convergence rates for Markov chains. *J. Amer. Statist. Assoc.* **90**, 558–566.

[19] Schassberger, R. (1978). Insensitivity of steady-state distributions of generalized semi-Markov processes with speeds. *Adv. Appl. Probab.* **10**, 836–851.

[20] Shedler, G. S. (1993). *Regenerative Stochastic Simulation*. Academic Press, New York.

[21] Sigman, K. (1990). One-dependent regenerative processes and queues in continuous time. *Math. Oper. Res.* **15**, 175–189.

[22] Thorisson, H. (2000). *Coupling, Stationarity, and Regeneration*. Springer-Verlag, New York.

[23] Whitt, W. (1980). Continuity of generalized semi-Markov processes. *Math. Oper. Res.* **5**, 494–501.