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Abstract

Koltun's arrangement method for linear programming is shown to be equivalent to using the simplex method to solve the standard so-called Phase I problem with artificial variables.

1 Introduction

In 2005 and 2006 Vladlen Koltun presented in various places his "Arrangement method for linear programming." A video of Koltun's talk on April 15, 2006 at the Bay Area Discrete Math Day XII is available at:
http://video.google.com/googleplayer.swf?docId=-6332244592098093013&hl=en.
A manuscript with the same title has been posted on the Web at:
http://theory.stanford.edu/~vladlen/lp.pdf.

Given $\mathbf{A} \in \mathbb{R}^{n\times d}$ and $\mathbf{b} \in \mathbb{R}^n$, the "arrangement method" [2] for finding $\mathbf{x} \in \mathbb{R}^d$ such that

$$\mathbf{A}\mathbf{x} \geq \mathbf{b} \tag{1}$$

amounts to a walk on the vertices of the arrangement $\mathcal{A}$ of the hyperplanes $H_i \equiv \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^T\mathbf{x} = b_i\}$ (where $\mathbf{a}_i^T$ is the ith row of $\mathbf{A}$). The walk is guided by a certain convex function $f : \mathbb{R}^d \to \mathbb{R}$, and it terminates at a minimum of f, which must be a feasible solution of (1) if there is one. Koltun constructed a highly degenerate so-called arrangement polytope and suggested that because the diameter of that polytope is bounded by a polynomial in n and d, this method might yield a strongly-polynomial algorithm for linear programming.

The standard "Phase I" method solves (1) as a linear minimization problem over a feasible domain:

$$\begin{aligned} \text{Minimize } & \mathbf{e}^T\mathbf{s} \\ \text{subject to } & \mathbf{A}\mathbf{x} + \mathbf{s} \geq \mathbf{b} \\ & \mathbf{s} \geq \mathbf{0} \end{aligned} \tag{2}$$

where $\mathbf{s} \in \mathbb{R}^n$ is a vector of so-called artificial variables and $\mathbf{e} = (1, \ldots, 1)^T \in \mathbb{R}^n$ (however, there is no need to introduce artificial variables for constraints that are satisfied at an initial point). The value of s_i at an optimal solution of (2) can be interpreted as the amount of violation of the constraint $\mathbf{a}_i^T\mathbf{x} \geq \mathbf{b}_i$, and the feasibility problem is solved by minimizing the total violation.

We show that the simplex method applied to (2) is equivalent to the arrangement method. Moreover, the diameter of the graph of feasible bases of (2) is polynomial.

2 The polyhedron of the Phase I method

Denote by P the feasible domain of (2), which is defined by $2n$ linear inequalities. For every $\mathbf{x} \in \mathbb{R}^d$ there exists an n-dimensional set of vectors $\mathbf{s}$ such that $(\mathbf{x}, \mathbf{s}) \in P$, and therefore,

FACT 2.1 *P is a polyhedron of dimension $d+n$ with $2n$ facets.*

PROPOSITION 2.1 *If $\mathbf{p} = (\mathbf{x}, \mathbf{s}) \in P$ is a vertex of P, then*

(*i*) *at least $d+n$ of the defining inequalities are tight at $\mathbf{p}$,*

(*ii*) *for every i, $i = 1, \ldots, n$,*

$$s_i = \max\{0, b_i - \mathbf{a}_i^T\mathbf{x}\} \ , \tag{3}$$

(*iii*) *for at least d values of i,*

$$s_i = b_i - \mathbf{a}_i^T\mathbf{x} = 0 \ .$$

For simplicity, we make the following nondegeneracy assumption:

ASSUMPTION 2.1 *Every submatrix $\mathbf{B} \in \mathbb{R}^{d\times d}$ of $\mathbf{A}$ is nonsingular and the basic solutions $\mathbf{x}(\mathbf{B}) \equiv \mathbf{B}^{-1}\mathbf{b}_B$ are distinct.*

Note that the $\mathbf{x}(\mathbf{B})$s are precisely the vertices of the arrangement $\mathcal{A}$.

DEFINITION 2.1 *We define a lifting mapping $\ell : \mathbb{R}^d \to \mathbb{R}^{d+n}$ as follows. For $\mathbf{x} \in \mathbb{R}^d$, $\ell(\mathbf{x}) = (\mathbf{x}, \mathbf{s})$, where $\mathbf{s}$ is defined by (3).*

PROPOSITION 2.2 *The vertices of P are precisely the points $\ell(\mathbf{x})$ where $\mathbf{x} = \mathbf{x}(\mathbf{B})$ for some B, i.e., there exist precisely d values of i for which $\mathbf{a}_i^T\mathbf{x} = b_i$.*

DEFINITION 2.2

(*i*) *A* line *of the arrangement $\mathcal{A}$ is a set of all $\mathbf{x}$ such that for some $d-1$ values of i, $\mathbf{a}_i^T\mathbf{x} = b_i$.*

(*ii*) *An (open)* edge *of $\mathcal{A}$ is a maximal open segment of a line of A where $\mathbf{a}_i^T\mathbf{x} = b_i$ for exactly $d-1$ values of i; if an edge is unbounded it is called a* ray.

PROPOSITION 2.3 *An open edge e of P is precisely a lifted set, $e = \ell(E)$, where E is an edge of $\mathcal{A}$.*

PROOF. If e is an edge connecting two vertices $\ell(\mathbf{x}(\mathbf{B}_1))$ and $\ell(\mathbf{x}(\mathbf{B}_2))$, then $|B_1 \cap B_2| = d-1$ and for every point $(\mathbf{x}, \mathbf{s})$ in the interior of e, for exactly $d-1$ values of i, namely, the intersection $B_1 \cap B_2$, $\mathbf{a}_i^T\mathbf{x} = b_i$. ☐

Corollary 2.1 *The lifting mapping ℓ maps vertices of $\mathcal{A}$ to vertices of P and edges of $\mathcal{A}$ to edges of P, preserving the vertex-edge incidence relationships; therefore, the diameter of P is equal to the diameter of $\mathcal{A}$.*

Corollary 2.2 *Every variant of the simplex method that walks over the vertices of P and decreases $\mathbf{e}^T\mathbf{s}$ monotonically, induces a walk on the vertices of $\mathcal{A}$, which monotonically decreases the total amount of violation of constraints given by*

$$\sum_{i=1}^{n} \max\{0, b_i - \mathbf{a}_i^T\mathbf{x}\} \ .$$

3 Comparison with Koltun's arrangement polytope

The "arrangement polyhedron" $\mathcal{AP}$ defined in [2] is $(d+1)$-dimensional and has exponentially-many facets corresponding to the d-dimensional cells of $\mathcal{A}$. Each vertex of $\mathcal{AP}$ is incident on 2^d facets and hence is highly degenerate as each vertex can be represented by many different bases. The Phase I polyhedron P is $(d+n)$-dimensional with only $2n$ facets and each vertex is incident only on $d+n$ facets; hence each vertex is represented by a unique basis.

References

[1] G. B. Dantzig, *Linear programming and extenstions,* Princeton University Press, Princeton, New Jersey, 1963.

[2] V. Koltun, "The arrangement method for linear programming," manuscript, January 5, 2007, http://theory.stanford.edu/~vladlen/lp.pdf