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Staffing at Multiple Locations for a Multi-Skill Service Provider

Ying Tat Leung
IBM Research Division
Almaden Research Center
650 Harry Road
San Jose, CA 95120-6099
USA

Jesse Bockstedt
School of Management
George Mason University
4400 University Drive, MS 1B1
Fairfax, VA 22030

Wolfgang König
Institute for Information Systems
Johann Wolfgang Goethe University
Mertsonstrasse 17
D-60325 Frankfurt am Main
Germany

Research Division
Almaden - Austin - Beijing - Cambridge - Haifa - India - T. J. Watson - Tokyo - Zurich

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Mertonstr. 17
D-60325 Frankfurt am Main
Germany

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Abstract

A common problem for firms that provide professional services requiring multiple skill types in a single project or in different types of projects is how to allocate employees to branch offices across multiple locations. Projects with varying skill requirements arise in these locations concurrently and multiple skills may be needed on a single project. The firm will lose projects if people of the correct skills are not available when a project arrives at a location. People can travel between locations to fill projects, but at some cost. Management consulting companies are typical examples of these service businesses. We develop a Markov model for a firm having employees with two skill types across two locations. Even for such a simple model, we resort to a numerical (but exact) solution. This model can be used as a baseline for comparison for more

general models in the future. Using the model we derive useful insights on the performance of the service business in terms of revenue, cost, and profit. Such insights are helpful to the development of staffing strategies for service providing organizations.

1. Introduction

As our economy continues to become increasingly more service-based (Chesbrough and Spohrer 2006), there is a growing need for new workforce management and staffing models. In service-based organizations human capital is typically the primary strategic resource, and effectively managing this resource will significantly impact attempts to maximize profits and minimize costs. Staffing and allocation of human resources is especially important for the project-oriented service organization. For example, revenue models for consulting services companies depend entirely on the efficient allocation of human resources. Employees that are staffed to projects result in revenue for the organization, while employees "riding the bench" (unassigned to a project) present a labor cost to the organization without commensurate revenue generation.

In the modern service organization, human resources may be assigned to projects or service calls across a large geographic area. This geographic expansion results in non-negligible travel and relocation costs typically comprised of transportation, lodging, and per diem reimbursements for the traveling employee. Additionally, firms in consulting and business services related industries frequently take on long term projects, often times several years in length, which obviously can incur significant travel costs. For firms with multiple office locations, one strategy to increase profits is to more effectively utilize local resources thus reducing travel and relocation costs. However, the stochastic nature of project assignments in the services industry makes optimal human resource planning a difficult problem. Indeed, the increasing need for flexible and efficient use of human resources in the modern organization (Powell 1990) requires the exploration of new models of human resource allocation (Rousseau 1995).

Incoming projects have several characteristics that make the prediction of staffing requirements difficult. First, the arrival rate of projects is stochastic. In other words, projects may arrive at any time and very seldom are the arrival of new project opportunities under the control of the consulting firm. Instead, the client organization submits requests for proposals or requests for quotes based on their own needs, which typically are not known ahead of time by the consulting services organization. Second, the length of projects is variable. Depending on the services required by a client organization, the length of a project may be as short as a couple of weeks or be a long term (several years) commitment. This variation in service commitment length increases the difficulty of planning available resources for incoming projects. Third, the required employee skill sets also vary across projects. Large consulting services companies such as Accenture or IBM tend to employ human resources across several skills, typically ranging from “soft” skills such as general management and marketing to technical skills such as accounting and language specific computer programming. Project opportunities may have comprehensive skill requirements across several disciplines or they may be more focused and require only a single skill. Fourth, the number of human resources required for each project is also variable. Projects of large scope may require the staffing of dozens if not hundreds of human resources to be effectively executed, while other projects may require as few as a single resource to meet the service needs of the client.

Considering these variable characteristics of modern projects, it is clear that effectively planning and predicting the staffing allocation needs of service-based organizations is a difficult problem. In this paper, we propose a new Markov model for analyzing the costs and revenue associated with the problem of staff allocation that considers the stochastic nature of modern service-based projects. Previous research in this vein has focused on deterministic optimization models such as linear programming, adopted ad hoc, heuristic-based approaches, or been non-quantitative. Surprisingly, there have been few models of workforce staffing that have adopted stochastic models or considered the role of travel costs in human resource allocation. Our model incorporates the non-deterministic nature of project arrivals and characteristics, and derives corresponding revenues and costs for a multi-location and multi-skill service provider.

Representing a first step to develop basic insights about such systems and about the challenges in developing models of such type, we build a model for a service provider operating from two locations and having two skills. Even for this simple situation, it turns out that analytical expressions for the revenue and cost of this service provider cannot be obtained and a numerical solution approach has to be used. Leveraging the computational sophistication of modern personal computers, we analyze several scenarios using our proposed model to derive basic insights, such as the effects of project characteristics and staffing strategies on the realized profits of the service provider. These insights are useful for developing human resource staffing strategies in a practical environment.

Our key conclusions from this effort are as follows.

1. On model development, we show that a Markov approach is a feasible vehicle to further our understanding of such service systems. In addition, since the solution to this model is exact, it can serve as a baseline for comparison in the development of more general but approximate

models. Such practice is well known and well accepted in the history of development of queueing models for manufacturing systems.

2. For a service provider that operates from two locations with two skills, our model shows that the staffing decision is not an easy one. Profit levels are generally not high and it seems that there is really no good solution – too many people will incur too high a cost and too few people will cause too many projects (and customers) to be lost. Our assumptions are admittedly rather stringent in some respects, but such a phenomenon will not be foreign to practicing service business managers. In addition, the performance (e.g., utilization of staff) estimated by a straightforward but rough calculation can be quite far from the actual performance, making the correct solution hard to obtain.
3. Using the model we analyze a commonly used strategy of centralization of skills, i.e., using centers of excellence in the organization. By computing a function of the tangible cost (travel cost) of skill centralization, any tangible and intangible benefits of such a structure can be traded off against this cost for it to be justified. The model essentially shows an efficient frontier of skill centralization.

The remainder of this paper proceeds as follows. Section 2 provides a brief overview of related research on workforce management, staffing, and human resource allocation. Section 3 presents our Markov model including the derived revenue and cost functions. Section 4 discusses several insights from the model including staffing strategies, corresponding revenue and costs derived from simulations, and the theoretical and practical implications of this research. Section 6 provides some concluding remarks.

2. Related Work

Prior research on workforce staffing and human resource management can be classified into two broad categories. First, many studies have developed qualitative frameworks that guide the human resource management role within an organization. The second broad category of research has focused on the development of mathematical models for optimizing workforce allocation across projects. The prior research in these two categories has largely been based in the context of construction management, software development, or general project-based industries. Professional service organizations might be considered a generalization of these industries. As such, we approach the problem of workforce staffing and allocation from this generalized viewpoint and believe the insights gained from our model can potentially be applied across many professional service industries.

Because of the strategic importance of human capital in a service based organization, the role of human resource management can be used to positively influence the satisfaction of customers (Bowen 1996). Therefore, much research has focused on effectively organizing and managing the human resource function in the service or project-based organization. Huemann et al. (2007) provide a good review of research on human resource management in a project-oriented organization, specifically focusing on consulting services firms. Lepak and Snell (1996) provide a general framework for human resource allocation that includes four different strategies: internal development, acquisition, contracting, and alliances. Hendricks et al. (1999) discuss the role of long-term, medium-term, and short-term allocation strategies on the effectiveness of human resource usage in a project-based research and development setting. In general, this stream of human resource allocation research has primarily focused on the development of useful but qualitative management frameworks for application in a service-based organization.

A separate stream of research complements the prior stream with the development of mathematical models for the human resource allocation problem. Several studies have modeled the resource allocation problem using a classical mathematical optimization or linear programming approach. Scheduling and production planning techniques are becoming more commonplace for the analysis and optimization of workforce allocation problems. For example, Gresh et al (2007) apply supply chain optimization techniques in the context of workforce planning. Gomer et al. (2002) developed a linear programming model to optimize the allocation of a multi-skilled workforce in construction projects. Using a more sophisticated approach, Mohring et al. (2003) incorporate minimal-cut calculations for assigning scarce resources to jobs in project scheduling. Other researchers have adopted constraint programming approaches for effectively assigning members of skilled workforce to projects (e.g., Naveh et al. 2007, Richter et al. 2008).

Recent approaches have also included mixed-integer (Adler and Hashai 2007) and game-theoretic techniques (Adler and Hashai 2008) for modeling the location-allocation problem and incorporating the intra and inter-firm flow of knowledge. Although the linear programming and constraint programming models provide useful methodologies for addressing the human resource planning problem, they typically assume a predefined set of projects.

We believe that a more strategic and more realistic representation of the scheduling and allocation process for the project-based service organization is a stochastic process of arriving projects. There has been some research that adopts a similar representation. For example, Schmidt (1996) applies a hybrid engineering / production Markov model to the management of research and development resources and for process design improvements. A series of models based on stochastic loss networks have also been developed to characterize the dynamics and uncertainty in the general management and allocation of a workforce in a service-based industry (Lu et al. 2006, Lu et al. 2007, Bharda et al. 2007).

The research presented in this paper continues this trend of incorporating the stochastic nature of project arrivals for a professional service organization. Specifically, we model the resource allocation process for a multi-location multi-skilled service provider using a Markov model and analyze several different staffing scenarios, as described next.

3. A Markov Model for Staffing

Consider a service business which operates at two locations, A and B, and possesses two key types of skills, 1 and 2. Each skill is assumed to be unique to a group of employees, so that there are two distinct groups of employees at each location. The business obtains revenue by working on projects that are initiated at either location. A project is initiated by a customer and requires a certain number of people of each skill. As is common in many types of service projects, the revenue of a project is proportional to the total number of person-unit time spent. We denote the revenue obtained from using skill 1(2) per unit time by R_1 (R_2). The business is free to assign people from either locations to work on a project, but there are travel costs involved if people at a remote location are assigned. This setting is common in a number of industries, from consulting to construction.

Projects arrive following an independent Poisson process with rates λ_A and λ_B at each corresponding location. At location A, there are N_{A1} and N_{A2} people with skill 1 and 2 respectively. Similarly, at location B, there are N_{B1} and N_{B2} people with skill 1 and 2 respectively. We assume that these people are permanent hires so that their cost is independent of whether they are working on projects. Let r_{A1} and r_{A2} respectively denote the total cost per unit time of employing one person of skill 1 and 2 at location A. For location B, r_{B1} and r_{B2} are used similarly. Each project requires a random number of people of skill 1 and 2, denoted by (m_1, m_2) where m_1 and m_2 are non-negative integers, and are determined when the project arrives with a joint probability $p(m_1, m_2)$.

When a project arrives at a location, the required number of people of each skill is immediately assigned to the project. People at the local location are always assigned first, with the remaining requirements assigned from the remote location if available. The travel or temporary relocation cost consists of two parts, a fixed cost per project per person denoted by s_{AB} (s_{BA}), and a variable cost per unit time per person denoted by s_{AB}' (s_{BA}') for people located at A(B) working on projects at B(A). If the total number of available people of either skill required by the project is not adequate, the project is assumed to be lost.

Once the project is assigned to people, all the required people will start working on the project. Each person will work on the project for an independent random duration following an exponential distribution with mean $1/\mu_i$, where i = 1 or 2, depending on the person's skill but regardless of the location of project arrival.

3.1. Performance Measures of the Service Business

We calculate the expected profit of the business using a Markov chain model. Let $n_{A1}(t)$ and $n_{A2}(t)$ represent the number of people of skill 1 and 2, respectively, working on a project from location A at time t; $n_{B1}(t)$ and $n_{B2}(t)$ represent the number of people of skill 1 and 2 from location B, respectively, working on a project at time t. Note that some of the people might have to travel to the other location for the project, but we only keep track of the number of people from a location that are busy, regardless of whether their projects are from the same or the other location. Consider the continuous time Markov chain with state $(n_{A1}(t), n_{A2}(t), n_{B1}(t), n_{B2}(t))$. Let $P(n_{A1}(t), n_{A2}(t), n_{B1}(t), n_{B2}(t))$ denote the probability of the state at time t and $\pi(n_{A1}, n_{A2}, n_{B1}, n_{B2})$ denote the steady-state probability as $t \rightarrow \infty$. For convenience, write $(n_{A1}, n_{A2}, n_{B1}, n_{B2})$ as $\underline{n}$ and $\pi(n_{A1}, n_{A2}, n_{B1}, n_{B2})$ as $\pi_{\underline{n}}$. Also, recall that $p(m_1, m_2)$, where $m_1 \geq 0$ and $m_2 \geq 0$, denote the probability that an arriving project (at either location) requires m_1 people of skill 1 and m_2 people of skill 2. For convenience later, we define $p(0, 0) = 0$.

Over a planning horizon of T, we have:

(3.1.1) E(gross profit) = E(revenue) – E(cost of fulfilling projects)

(3.1.2) E(revenue) = E(revenue of all incoming projects – revenue of lost projects)

(3.1.3) E(revenue of all incoming projects) =

$$T \sum_{m_1, m_2} (m_1 R_1 / \mu_1 + m_2 R_2 / \mu_2)(\lambda_A + \lambda_B) p(m_1, m_2)$$

Let i_{A1}, i_{A2}, i_{B1}, i_{B2}, i_1, i_2 be non-negative integers, and

$V = \{(i_{A1}, i_{A2}, i_{B1}, i_{B2}) \mid i_{A1} \leq N_{A1}, i_{A2} \leq N_{A2}, i_{B1} \leq N_{B1}, i_{B2} \leq N_{B2}\}$,

$W(\underline{n}) = \{(i_1, i_2) \mid i_1 > N_{A1} + N_{B1} - n_{A1} - n_{B1}\} \cup \{(i_1, i_2) \mid i_2 > N_{A2} + N_{B2} - n_{A2} - n_{B2}\}$. V is the state space of the Markov chain and $W(\underline{n})$ represents the set of incoming projects that are beyond the currently available capacity of the business. Then

(3.1.4) E(revenue of lost projects) =

$$\sum_{\underline{n} \in V} \left[\pi_{\underline{n}} T \sum_{(m_1, m_2) \in W(\underline{n})} (m_1 R_1 / \mu_1 + m_2 R_2 / \mu_2)(\lambda_A + \lambda_B) p(m_1, m_2) \right]$$

Using equations (3.1.2), (3.1.3), and (3.1.4), we obtain E(revenue).

We now turn to the cost of fulfilling projects with our staff. First, we have

(3.1.5) E(cost of fulfilling projects) = E(labor cost) + E(travel cost)

Labor cost is straightforward since we assume that all employees are permanent hires and the cost is therefore fixed.

(3.1.6) E(labor cost) =

$$(r_{A1}N_{A1} + r_{B1}N_{B1} + r_{A2}N_{A2} + r_{B2}N_{B2})T$$

To model some level of economy of scale for people of the same skill being in the same location, we let r_{A1} to be a non-increasing function of N_{A1}, r_{A2} to be a non-increasing function of N_{A2}, etc. This is true, for example, if people of the same skill can share equipment, tools, or knowledge acquisition activities, or can be managed by a single administration staff.

Let j_1 and j_2 be non-negative integers, and

$W'(\underline{n}) = \{(j_1, j_2) \mid 0 \le j_1 \le N_{A1} + N_{B1} - n_{A1} - n_{B1},\ 0 \le j_2 \le N_{A2} + N_{B2} - n_{A2} - n_{B2}\}$.

$W'(\underline{n})$ represents the set of projects arriving when the Markov chain is in state $\underline{n}$ and that can be accepted by the business. When a project is accepted at a location, the number of people of a particular skill who have to travel is equal to the number required that is beyond those currently available at that location. The fixed cost of travel is this number multiplied by the unit fixed cost (s_{BA} or s_{AB}), while the variable cost is this number multiplied by the expected time required of a skill and the unit variable travel cost (s_{BA}' or s_{AB}'). Hence the total travel cost is as follows.

(3.1.7) E(travel cost) =

$$\sum_{\underline{n}\in V}\pi_{\underline{n}}\sum_{(m_1,m_2)\in W'(\underline{n})} p(m_1,m_2)\Big\{\lambda_A T s_{BA}\left[(m_1-(N_{A1}-n_{A1}))^+ + (m_2-(N_{A2}-n_{A2}))^+\right]+$$

$$\lambda_A T s_{BA}'\left[(m_1-(N_{A1}-n_{A1}))^+/\mu_1 + (m_2-(N_{A2}-n_{A2}))^+/\mu_2\right]+$$

$$\lambda_B T s_{AB}\left[(m_1-(N_{B1}-n_{B1}))^+ + (m_2-(N_{B2}-n_{B2}))^+\right]+$$

$\lambda_B Ts_{AB}'\left[(m_1-(N_{B1}-n_{B1}))^+/\mu_1+(m_2-(N_{B2}-n_{B2}))^+/\mu_2\right]\}$

Using equations (3.1.5), (3.1.6), and (3.1.7), we obtain E(cost of fulfilling projects).

3.2. The Transition Rate Matrix of the Markov Chain

In this section, we develop the transition rate matrix of the Markov chain ($n_{A1}(t)$, $n_{A2}(t)$, $n_{B1}(t)$, $n_{B2}(t)$) so that the steady-state probabilities of all its states can be calculated.

Consider a state $\underline{n} = (j_{A1}, j_{A2}, j_{B1}, j_{B2})$ where $(j_{A1}, j_{A2}, j_{B1}, j_{B2}) \in V$. For convenience, we write $(j_{A1}, j_{A2}, j_{B1}, j_{B2})$ as $\underline{j}$. Similarly, we denote another state different from $\underline{j}$, $(i_{A1}, i_{A2}, i_{B1}, i_{B2}) \in$ V as $\underline{i}$. Let Q be the transition rate matrix for the Markov chain and let the elements of Q be indexed by the state vector, i.e., $Q = [q_{\underline{ij}}]$. Consider a given state $\underline{j}$ and the transitions resulting in $\underline{j}$. Possible state transitions correspond to the following events:

1. One person, at either location and of either skill, completes his part of a project;
2. A new project arrives at location A;
3. A new project arrives at location B.

Depending on the value of $\underline{j}$, one or more of the above transitions can lead to $\underline{j}$. Take, for example, the case of $\underline{j}$ in the interior of V, i.e., $0 < j_{A1} < N_{A1}$, $0 < j_{A2} < N_{A2}$, $0 < j_{B1} < N_{B1}$, $0 < j_{B2} < N_{B2}$. In this case, $\underline{j}$ can come from all three transitions. For transitions representing project completion by a person, the rate is $i_{A1}\mu_1$ if the previous state $\underline{i}$ is $(j_{A1} + 1, j_{A2}, j_{B1}, j_{B2})$, $i_{A2}\mu_2$ if the previous state $\underline{i}$ is $(j_{A1}, j_{A2} + 1, j_{B1}, j_{B2})$, etc. For project arrivals at either A or B, we know that the arrival did not result in anyone traveling because $\underline{j}$ is still under capacity for each location-skill. So if a project arrived at A, we know that its requirements must have been $(j_{A1} - i_{A1}, j_{A2} - i_{A2})$. Similarly, if a project arrived at B, its requirements must have been $(j_{B1} - i_{B1}, j_{B2} - i_{B2})$. This is illustrated in Figure 3.1 where we have drawn state space V in three dimensions, instead of the correct, four dimensions.

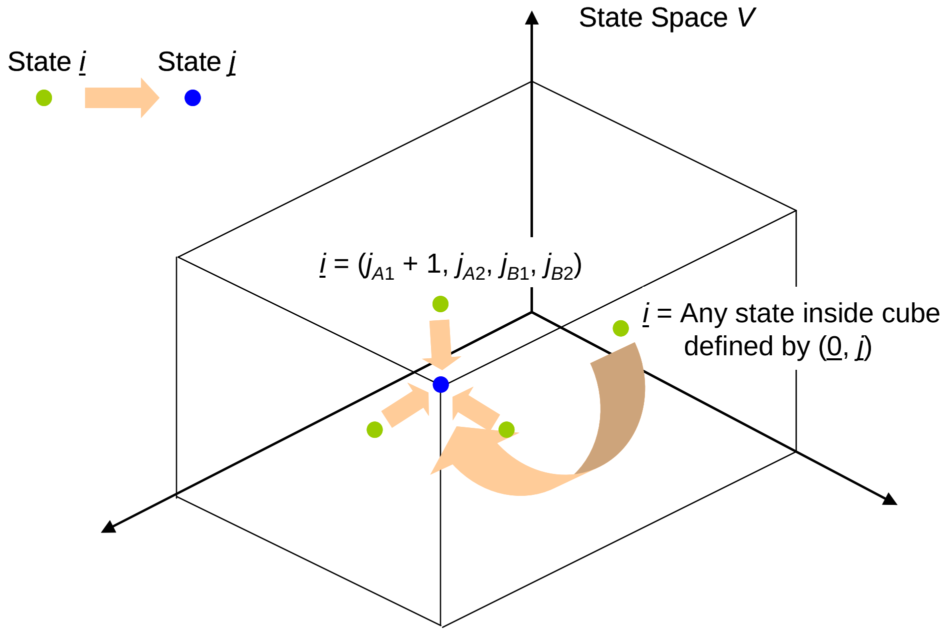

Figure 3.1. Conceptual Illustration of State Transitions from $\underline{i}$ to $\underline{j}$

In mathematical notation, we have

$$q_{\underline{ij}} = i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + I(\underline{i}\in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i}\in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2})$$

where

$$I_{A1}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}+1, i_{A2} = j_{A2}, i_{B1} = j_{B1}, i_{B2} = j_{B2}, \underline{i}\in V, \underline{j}\in V \\ 0, & \text{otherwise} \end{cases}$$

$$I_{A2}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}, i_{A2} = j_{A2}+1, i_{B1} = j_{B1}, i_{B2} = j_{B2}, \underline{i}\in V, \underline{j}\in V \\ 0, & \text{otherwise} \end{cases}$$

$$I_{B1}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}, i_{A2} = j_{A2}, i_{B1} = j_{B1}+1, i_{B2} = j_{B2}, \underline{i}\in V, \underline{j}\in V \\ 0, & \text{otherwise} \end{cases}$$

$$I_{B2}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}, i_{A2} = j_{A2}, i_{B1} = j_{B1}, i_{B2} = j_{B2} + 1, \underline{i} \in V, \underline{j} \in V \\ 0, & \text{otherwise} \end{cases}$$

$$L_A(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 \le n_{A1}, a_2 \le n_{A2}, b_1 = n_{B1}, b_2 = n_{B2}\}$$

$$L_B(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 = n_{A1}, a_2 = n_{A2}, b_1 \le n_{B1}, b_2 \le n_{B2}\}$$

a_1, a_2, b_1, b_2 are non-negative integers.

If $\underline{j}$ is at an edge of V, say $j_{A1} = N_{A1}$, then there is no transition due to project completion by skill 1 at A and there might be people of skill 1 traveling from B to A when a project arrived at A. The rest of the situation is similar to the case above. Following this reasoning, we can derive the transition rates to $\underline{j}$ for each possible position of $\underline{j}$ in V. There are a total of 16 cases, the details of which are contained in the Appendix. All 16 cases can be consolidated into a single, albeit somewhat complicated, expression for $q_{\underline{ij}}$ for $i \neq j$, and one for $\underline{i} = \underline{j}$. They are listed as equations (A.17) and (A.18) respectively in the Appendix.

3.3. Solving the Model

To solve the model, we first solve for the steady-state probabilities of all the states of the Markov chain, $\pi_{\underline{n}}$, then use $\pi_{\underline{n}}$ in equations (3.1.1) – (3.1.7) to compute the performance measures.

Let $\pi = (\pi_{\underline{n}})$ be the vector of stationary probabilities of the Markov chain, indexed by the state vector $\underline{n}$. We need to solve the following set of simultaneous linear equations for π.

(3.3.1) $\sum_{\underline{i} \in V} \pi_{\underline{i}} = 1$,

$\pi Q = 0$, i.e., $\forall \underline{j} \in V, \quad \sum_{\underline{i} \in V} \pi_{\underline{i}} q_{\underline{ij}} = 0$, where q_{ij} is given by equations (A.17) and (A.18) in the Appendix.

It seems impossible to obtain a closed-form solution to (3.3.1); we therefore have to settle for a numerical solution. A computer program was written to generate the Q matrix and then the system (3.3.1) is solved numerically. Many existing software packages are possible for the latter, and we have chosen to use GNU Octave (available at http://www.gnu.org/software/octave/), an open source software package for numerical computations. Based on its documentation, for the type of system defined in (3.3.1), it uses a GR Factorization algorithm. The solution for π in (3.3.1) is then used in another program to calculate the performance measures defined in section 3.1.

In the entire solution computation process, the step of solving system (3.3.1) is the most computationally intensive. For small but non-trivial systems such as those discussed in Section 4, the time needed to solve (3.3.1) using Octave on a Windows PC with an Intel Core 2 Duo 2.4 GHz processor and 2 GB RAM is in the order of seconds to tens of seconds. This is achieved without any attempt to optimize the computation process.

4. Some Insights from the Model

In this section, we use the model to gain some insights into the impact of different design decisions or environmental factors on the performance of such a service business. We study three different scenarios to investigate the effect of the level of staffing, the variance of the staffing requirements of arriving projects, and the effect of centralized or distributed staffing strategies.

Unless otherwise noted, we use the following parameter settings of the model. Although the absolute numbers used are hypothetical, their ratios are designed to be representative of at least one service business (consulting).

Project arrival rates: $\lambda_B = 2\ \lambda_A$, or $\lambda_A = \lambda_B$

Project completion rates per person: $\mu_1 = 0.6\ \lambda_A$, $\mu_2 = \lambda_A$

Distribution of project staffing requirements, i.e., $p(m_1, m_2)$ is uniform over the square spanning (0, 0) to (3, 3) (except for an adjustment for number rounding) with $p(0, 0) = 0$. Table 4.0.1 contains the exact probabilities used.

Labor cost rates: $r_{A1} = r_{B1}$, $r_{A2} = r_{B2}$, $r_{A1} = 0.75\ r_{A2}$

Travel cost rates: $s_{AB} = s_{BA} = r_{A1}/5$, $s_{AB}' = s_{BA}' = r_{A1}/3$

Revenue rates: $R_1 = 2\ r_{A1}$, $R_2 = 2\ r_{A2}$

Table 4.0.1. Probabilities of Staffing Requirements of an Incoming Project

$p(m_1, m_2)$		m_2			
		0	1	2	3
m_1	0	0	0.0666	0.0666	0.0666
	1	0.0666	0.0666	0.0666	0.0666
	2	0.0666	0.0666	0.0666	0.0666
	3	0.0666	0.0666	0.0666	0.0676

4.1. Level of Staffing

In this scenario, we set $\lambda_A = 1$ and choose $\lambda_B = 2\ \lambda_A$, $r_{A1} = 6000$. To see the impact of staffing level alone as much as possible, we vary the number of people of each skill at each location uniformly, i.e., we use the range of (N_{A1}, N_{A2}, N_{B1}, N_{B2}) being (1, 1, 1, 1) to (6, 6, 6, 6).

In addition to the performance measures such as revenue, profit, and the several kinds of costs defined in Section 3.1, we also calculate the following quantity which is a useful reference:

Max revenue = the maximum revenue that can be realized by the business, which is equal to the revenue when every person of every skill is working on projects all the time.

Figure 4.1.1 shows the effect of staffing level on revenue. In the figure, the labels on the x-axis are in the form of N_{A1}-N_{A2}-N_{B1}-N_{B2}; Lost Revenue is calculated by equation (3.1.4), Actual Revenue by equation (3.1.2), Potential Revenue by equation (3.1.3). Potential Revenue is constant, since the amount of incoming projects is independent of how the business is staffed. Max Revenue grows linearly as N_{A1}-N_{A2}-N_{B1}-N_{B2} increases by 1-1-1-1, since we have a fixed revenue rate per person-time unit. Lost Revenue decreases as the total number of people increases at a decreasing rate. Actual Revenue is the difference between Potential Revenue and Lost Revenue, and is therefore approaching Potential Revenue at a decreasing rate. This is as expected of a stochastic, queuing-like system.

A simplistic way of planning for staffing is to use a "planned utilization" of the staff which is (Potential Revenue)/(Max Revenue). This of course ignores the dynamics of the system but is straightforward to calculate. Max Revenue can be computed with simple arithmetic and Potential Revenue can be estimated from recent historical data. It is tempting as a rough measure for high level planning and is indeed what a static model will provide. On the other hand, the correct or actual utilization of the staff is (Actual Revenue)/(Max Revenue). Table 4.1.1 shows the planned and actual utilizations as we vary N_{A1}-N_{A2}-N_{B1}-N_{B2} corresponding to that in Figure 4.1.1. When N_{A1}-N_{A2}-N_{B1}-N_{B2} = 1-1-1-1, it appears that the system is under-capacity since the amount of incoming projects is more than three times what we have time for. It would therefore be not unreasonable to expect a fairly high actual utilization. But in fact we lose almost

60% of the potential revenue due to the dynamics of the system and end up with a utilization of 41.8%. As the number of staff increases, the lost revenue decreases and the planned utilization becomes closer to the actual utilization, as shown in Table 4.1.1. However, while the planned utilization predicts a wide range of 50% to 100% (naturally cutting off all figures at 100%), the actual utilization shows a relatively small range of 42% to 54%. In addition, the actual utilization displays a maximum at 3-3-3-3, which the simplistic, linear planned utilization would never be able to predict.

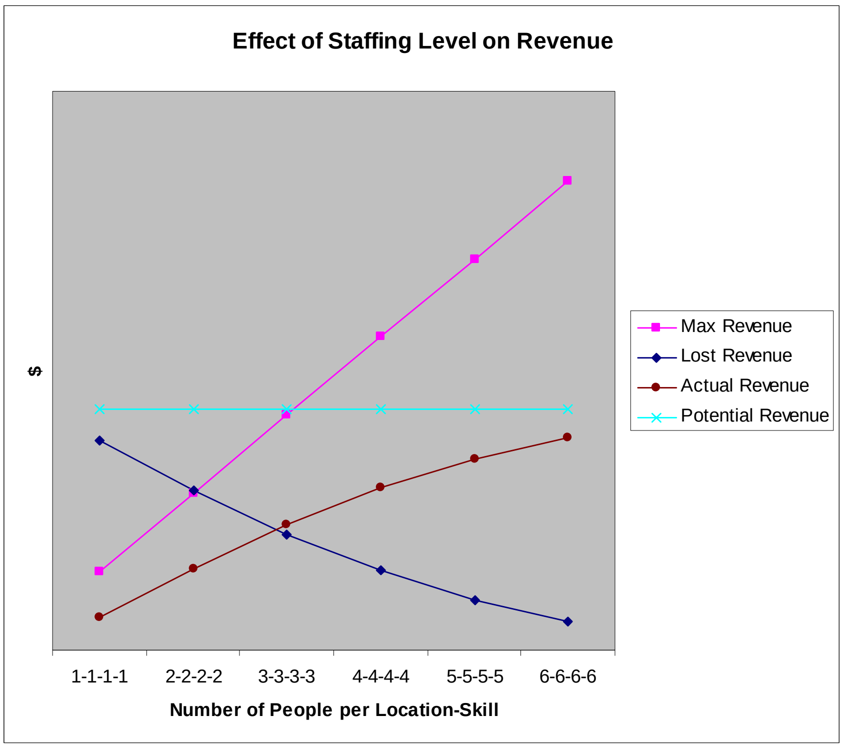

Figure 4.1.1. Effect of Staffing Level on Revenue

Table 4.1.1. Planned and Actual Utilization of Staff

Number of People Per Location-Skill	Planned Utilization	Actual Utilization	Lost Revenue as % of Actual Revenue
1-1-1-1	308.8%	41.8%	639.4%
2-2-2-2	154.4%	52.3%	195.4%
3-3-3-3	102.9%	53.5%	92.6%
4-4-4-4	77.2%	51.9%	48.8%
5-5-5-5	61.8%	49.0%	26.1%
6-6-6-6	51.5%	45.3%	13.6%

Figure 4.1.2 shows the effect of staffing level on the labor cost (using equation 3.1.6), travel cost (using equation 3.1.7), and the profit (using equation 3.1.1) of the business. For reference, we have plotted the Actual Revenue again in the same figure. (Note that the quantities may be plotted on different scales even though they are on the same graph. For example, the labor cost and travel cost are on different scales; Labor Cost is typically over 90% of Actual Revenue while Travel Cost accounts for 5-10% only.) The labor cost rises linearly as N_{A1}-N_{A2}-N_{B1}-N_{B2} increases by 1-1-1-1, as evident from equation (3.1.6). Travel Cost has a concave shape, representing the following phenomenon. A very small number of people naturally limits the total amount of travel cost. The cost increases as there are more people available to travel. On the other hand, a larger number of people per location-skill reduces the need to travel, since the project requirements can be handled locally at the project arrival location. Profit is the difference between the Actual Revenue (which is almost linear over the selected range of people-skill) and the sum of Labor Cost (linear) and Travel Cost (concave).

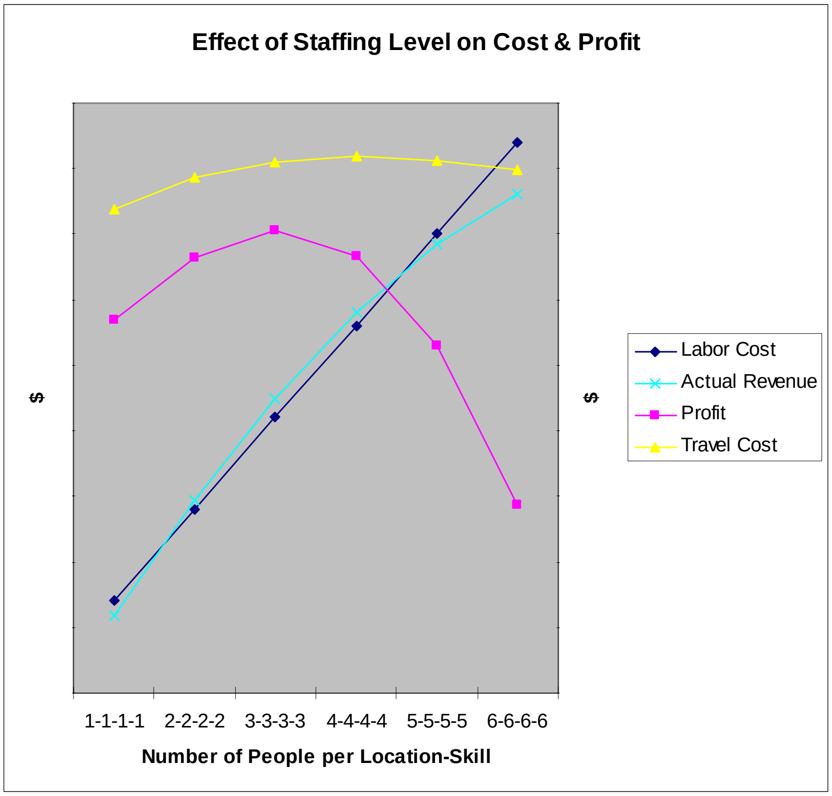

Figure 4.1.2. Effect of Staffing Level on Costs and Profit

4.2. Variance of Project Requirements

In this scenario, we use the same parameter settings as those in Section 4.1, except that the variance of project staffing requirements is increased by extending the distribution to cover from (0, 0) to (6, 6). In order to maintain marginal expectations $E(m_1)$ and $E(m_2)$ the same as before, we solve a simple optimization problem to obtain the probabilities. Table 4.2.1 shows these probabilities after adjusting for number rounding.

Table 4.2.1. Probabilities of Staffing Requirements of an Incoming Project

$p(m_1, m_2)$		m_2						
		0	1	2	3	4	5	6
m_1	0	0	0.22797	0.01892	0.0188	0.01868	0.01856	0.001
	1	0.22797	0.02314	0.02073	0.02062	0.0205	0.02038	0.001
	2	0.01892	0.02073	0.02092	0.02243	0.02231	0.001	0.001
	3	0.0188	0.02062	0.02243	0.01869	0.001	0.001	0.001
	4	0.01868	0.0205	0.02231	0.001	0.01647	0.001	0.001
	5	0.01856	0.02038	0.001	0.001	0.001	0.01852	0.001
	6	0.001	0.001	0.001	0.001	0.001	0.001	0.02246

Figures 4.2.1 and 4.2.2 show the revenues and costs/profit with the new $p(m_1, m_2)$ given in Table 4.2.1. Comparing to Figures 4.1.1 and 4.1.2, we can see that the basic behavior of the business remains the same. Except for the case of $(N_{A1}, N_{A2}, N_{B1}, N_{B2}) = (1, 1, 1, 1)$, the Actual Revenue and Profit are generally lower than the previous case due to the higher Revenue Lost and higher Travel Cost, despite identical $E(m_1)$ and $E(m_2)$ values. The Actual Revenue also approaches Potential Revenue at a lower rate. The (1, 1, 1, 1) case seems to be anomaly because the probability of having $(m_1, m_2) = (1, 0)$ or $(0, 1)$ is higher than the low variance case in order to maintain the same means. The higher profit of the (1, 1, 1, 1) case is also verified by its higher actual utilization of staff, as seen when comparing Table 4.2.1 and Table 4.1.1. (Utilization drives profit to a large extent in these examples because labor cost is fixed and travel cost is significantly lower than labor cost.)

The point of maximum profit shifts to a smaller value of number of people per location-skill. In fact, it occurs at the smallest value of the selected set. This is because the rate of increase of Actual Revenue is smaller than the linear rate of increase of Labor Cost which is a major component of the total cost. Intuitively, it seems that the high variance in project staffing requirements causes many projects to be lost even at a fairly large staff capacity and so the highest profit is to be obtained by a lowest cost, i.e. having a smallest capacity.

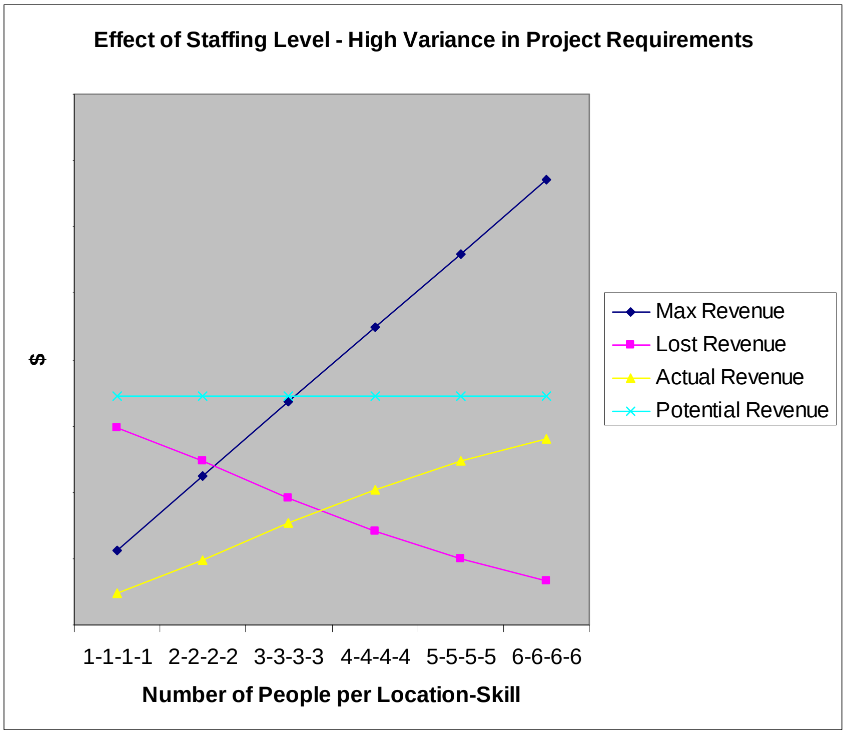

Figure 4.2.1. Effect of Staffing Level on Revenue for the Case of High Variance in Project Staffing Requirements

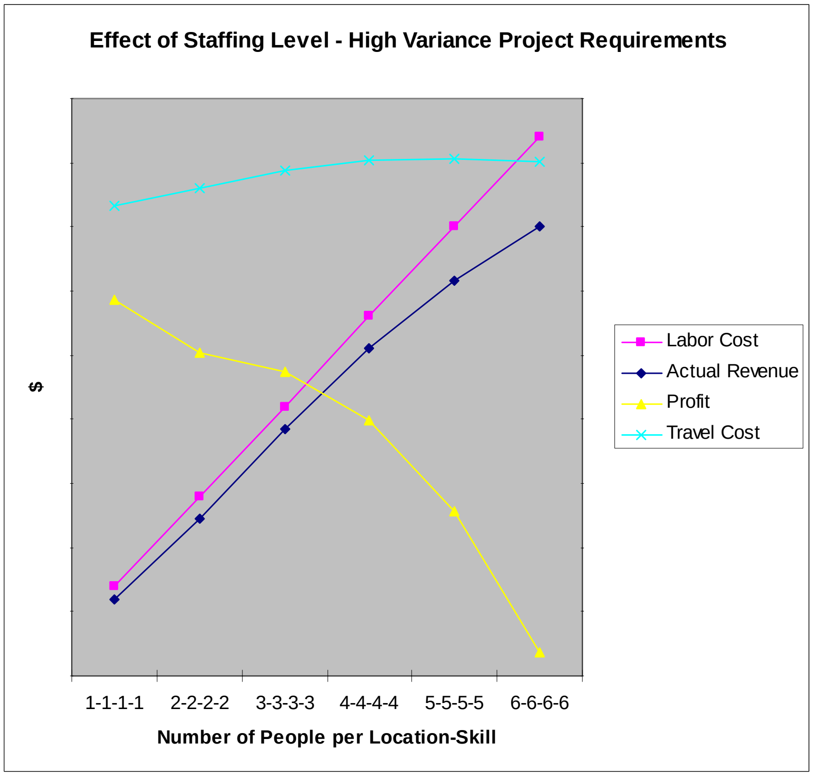

Figure 4.2.2. Effect of Staffing Level on Costs and Profit for the Case of High Variance in Project Staffing Requirements

Table 4.2.1. Planned and Actual Utilization of Staff for the Case of High Variance in Project Staffing Requirements

Number of People Per Location-Skill	Planned Utilization	Actual Utilization	Lost Revenue as % of Actual Revenue
1-1-1-1	308.9%	42.7%	623.1%
2-2-2-2	154.4%	43.9%	252.0%
3-3-3-3	103.0%	45.9%	124.4%
4-4-4-4	77.2%	45.6%	69.5%
5-5-5-5	61.8%	44.0%	40.4%
6-6-6-6	51.5%	41.7%	23.6%

4.3. Centralization of Skills

In manufacturing industries where customer demand is filled with inventory in stock, the issue of inventory centralization has been well studied, starting with Eppen (1979). Here we study the analogous issue of skill centralization in service industries where customer demand is filled with the labor time of staff.

Conventional wisdom suggests that when staff of the same skill is centralized in a single location, there will be cost advantages due to economies of scale. For example, the staff can share equipment, materials and parts (i.e., inventory centralization), libraries of technical literature and past project documentation (although library issues are decreasingly important with electronic documentation), people management and facility costs. There will also be intangible benefits such as people sharing their work experience informally and the chance for face-to-face discussions and problem solving. The latter is particularly important for creative work or complex technical. On the other hand, a decentralized staffing approach where skills are replicated in different locations will potentially lead to the cost advantage of lesser amount of travel to customer locations and intangible benefits that include a higher level of familiarity of local customers and local business climate of the staff, and a higher potential for cross-skill idea generation.

For some industries, notably manufacturing of mass production type, the tangible benefits of centralization are dominant and the existing pattern of centralized locations can confirm. Service industries are usually much less resource based so that economies of scale in resource usage will not dominate. In general, the overall effect of skill centralization is not clear. Quantitative studies of this issue are seldom seen, yet a commonly used strategy by many service businesses and non-manufacturing functions within a manufacturing business in the last two decades is the so-called center of excellence approach, where few related skills are centralized to serve all customers or the needs of the entire organization (Waller 1998).

Intangible benefits of skill centralization are difficult to estimate and represent a fruitful research subject in itself. Therefore, we restrict ourselves to analyzing the tangible tradeoffs, the major source of which is cost. Once the cost tradeoff is estimated, a subjective judgment can be exercised to determine whether the net intangible benefits are greater than the total cost saving.

To focus on the effect of skill centralization, we set the project arrival rates at the two locations of our model to be identical and run the model for a range of scenarios that spans the two extremes of a completely centralized case (N_{A1}, N_{A2}, N_{B1}, N_{B2}) = (10, 0, 0, 10) or (0, 10, 10, 0) to a completely decentralized case (N_{A1}, N_{A2}, N_{B1}, N_{B2}) = (5, 5, 5, 5). The latter is effectively two identical businesses set up in the two locations with "emergency" travel between the two businesses in case of capacity overflow. It is clear that the Max, Potential, Lost, and Actual Revenues, and the Labor Cost are identical across these cases. The Travel Cost is the determinant of the Profit, as shown in Figure 4.3.1. As expected, the maximum and minimum Travel Cost occurs at the two extremes of completely centralized (10, 0, 0, 10) and decentralized (5, 5, 5, 5) cases.

Recall that our model assumes no change in all cost rates for all the cases, i.e., there is no benefit of centralization or decentralization. The difference in Profit (or Travel Cost) between the extremes, denoted by w in Fig. 4.3.1, is the least amount of centralization benefit for a center-of-excellence approach to be justified. This benefit would include the sum of all tangible and intangible benefits, as discussed before. Indeed, the graph of Profit ranging from (10, 0, 0, 10) to (5, 5, 5, 5) represents an efficient frontier for skill centralization over the different operating points. This is highlighted in Figure 4.3.1.

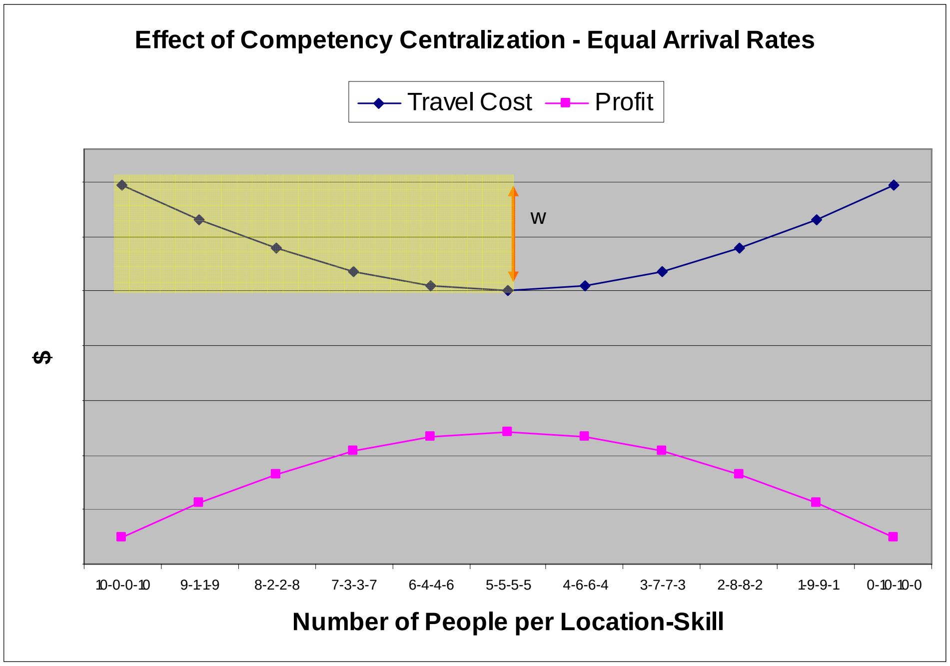

Figure 4.3.1. Effect of Skill Centralization on Travel Cost and Profit for the Case of Equal Project Arrival Rates at the Two Locations

5. Concluding Remarks

This paper represents a first step in quantitatively understanding the impact of staffing decisions at multiple locations on revenue, costs, and profit for a service business. We utilize a simple, Markov model with two locations to draw insights on the behavior of the system and on the challenges of developing a stochastic model of such systems. Even for such a simple model, an analytical expression for the profit of the business seems impossible and a numerical but exact solution for this model is derived. This exact solution can be used as a baseline for comparison for more general but approximate models in the future. Researchers interested in such usage can contact the authors for more extensive numerical results from the present model. We also demonstrate some of the insights gained through analyzing three scenarios using the model. Such insights are helpful to the development of staffing strategies for service providing organizations.

It turns out that the numerical solution to the model using standard PC hardware and open source software for small models is quite fast, so that an optimal solution can be obtained using a straightforward, exhaustive search over a reasonable set of staffing levels. More intelligent approaches, such as exploiting useful properties of the model for optimization, are likely possible. This is a subject of our future research.

Many other extensions and relaxations of the model will be interesting and will enable a closer approximation to reality. Some of the more important ones include the following.

1. Projects may not be lost immediately if the current staff capacity is inadequate. Customers are usually willing to wait for some time period before they take the business elsewhere.
2. The travel time between locations is not zero. This will consume some of the productive capacity of the staff. The fixed cost in the current model reflects some aspect of this.
3. The distribution of the work duration of each person for a project is non-exponential. For example, one can use a phase-type distribution which will keep the model Markovian but will further complicate the model algebraically.

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Appendix: The Transition Rates of the Markov Chain

Let a_r, b_r, $r = 1, 2$, be non-negative integers. Define

$$L_A(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 \le n_{A1}, a_2 \le n_{A2}, b_1 = n_{B1}, b_2 = n_{B2}\}$$

$$L_B(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 = n_{A1}, a_2 = n_{A2}, b_1 \le n_{B1}, b_2 \le n_{B2}\}$$

$$L_{A1}(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 \le n_{A1}, a_2 \le n_{A2}, b_1 < n_{B1}, b_2 = n_{B2}\}$$

$$L_{A2}(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 \le n_{A1}, a_2 \le n_{A2}, b_1 = n_{B1}, b_2 < n_{B2}\}$$

$$L_{B1}(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 < n_{A1}, a_2 = n_{A2}, b_1 \le n_{B1}, b_2 \le n_{B2}\}$$

$$L_{B2}(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 = n_{A1}, a_2 < n_{A2}, b_1 \le n_{B1}, b_2 \le n_{B2}\}$$

$$L_{A1A2}(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 \le n_{A1}, a_2 \le n_{A2}, b_1 < n_{B1}, b_2 < n_{B2}\}$$

$$L_{B1B2}(n_{A1},n_{A2},n_{B1},n_{B2}) = \{(a_1,a_2,b_1,b_2) \mid a_1 < n_{A1}, a_2 < n_{A2}, b_1 \le n_{B1}, b_2 \le n_{B2}\}$$

Define the indicator function of a relation S by

$$I(S) = \begin{cases} 1, & S = \text{True} \\ 0, & \text{otherwise} \end{cases}$$

Further, define

$$I_{A1}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}+1, i_{A2} = j_{A2}, i_{B1} = j_{B1}, i_{B2} = j_{B2}, \underline{i} \in V, \underline{j} \in V \\ 0, & \text{otherwise} \end{cases}$$

$$I_{A2}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}, i_{A2} = j_{A2}+1, i_{B1} = j_{B1}, i_{B2} = j_{B2}, \underline{i} \in V, \underline{j} \in V \\ 0, & \text{otherwise} \end{cases}$$

$$I_{B1}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}, i_{A2} = j_{A2}, i_{B1} = j_{B1} + 1, i_{B2} = j_{B2}, \underline{i} \in V, \underline{j} \in V \\ 0, & \text{otherwise} \end{cases}$$

$$I_{B2}(\underline{i},\underline{j}) = \begin{cases} 1, & i_{A1} = j_{A1}, i_{A2} = j_{A2}, i_{B1} = j_{B1}, i_{B2} = j_{B2} + 1, \underline{i} \in V, \underline{j} \in V \\ 0, & \text{otherwise} \end{cases}$$

We now calculate the transition rate going into state $\underline{j}$, depending on the value of $\underline{j}$, as in the following 16 cases.

Case (i): $j_{A1} < N_{A1}, j_{A2} < N_{A2}, j_{B1} < N_{B1}, j_{B2} < N_{B2}$. In this case, there was no assignment of any remote skill in reaching state $\underline{j}$.

(A.1)
$$\begin{aligned} q_{\underline{ij}} = \; & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\ & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) \end{aligned}$$

Case (ii): $j_{A1} = N_{A1}, j_{A2} < N_{A2}, j_{B1} < N_{B1}, j_{B2} < N_{B2}$. In this case, skill 1 at B might have been assigned to A in reaching state $\underline{j}$.

(A.2)
$$\begin{aligned} q_{\underline{ij}} = \; & i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\ & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, j_{A2} - i_{A2}) \end{aligned}$$

Case (iii): $j_{A1} < N_{A1}, j_{A2} = N_{A2}, j_{B1} < N_{B1}, j_{B2} < N_{B2}$. In this case, skill 2 at B might have been assigned to A in reaching state $\underline{j}$.

(A.3)
$$\begin{aligned} q_{\underline{ij}} = \; & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\ & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{A2}(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) \end{aligned}$$

Case (iv): $j_{A1} < N_{A1}, j_{A2} < N_{A2}, j_{B1} = N_{B1}, j_{B2} < N_{B2}$. In this case, skill 1 at A might have been assigned to B in reaching state $\underline{j}$.

(A.4)
$$\begin{aligned} q_{\underline{ij}} = \; & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\ & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{B1}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, j_{B2} - i_{B2}) \end{aligned}$$

Case (v): $j_{A1} < N_{A1}, j_{A2} < N_{A2}, j_{B1} < N_{B1}, j_{B2} = N_{B2}$. In this case, skill 2 at A might have been assigned to B in reaching state $\underline{j}$.

$$
\text{(A.5)} \quad
\begin{aligned}
q_{\underline{ij}} = \; & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, N_{B2}-i_{B2}+j_{A2}-i_{A2})
\end{aligned}
$$

Case (vi): $j_{A1} = N_{A1}, j_{A2} = N_{A2}, j_{B1} < N_{B1}, j_{B2} < N_{B2}$. In this case, either one of or both skill 1 and 2 at B might have been assigned to A in reaching state $\underline{j}$.

$$
\text{(A.6)} \quad
\begin{aligned}
q_{\underline{ij}} = \; & i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}+j_{B1}-i_{B1}, N_{A2}-i_{A2}) + \\
& I(\underline{i} \in L_{A2}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}, N_{A2}-i_{A2}+j_{B2}-i_{B2}) + \\
& I(\underline{i} \in L_{A1A2}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}+j_{B1}-i_{B1}, N_{A2}-i_{A2}+j_{B2}-i_{B2})
\end{aligned}
$$

Case (vii): $j_{A1} = N_{A1}, j_{A2} < N_{A2}, j_{B1} = N_{B1}, j_{B2} < N_{B2}$. In this case, either skill 1 at B might have been assigned to A or skill 1 at A might have been assigned to B in reaching state $\underline{j}$.

$$
\text{(A.7)} \quad
\begin{aligned}
q_{\underline{ij}} = \; & i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}+N_{B1}-i_{B1}, j_{A2}-i_{A2}) + \\
& I(\underline{i} \in L_{B1}(\underline{j}))\lambda_B p(N_{B1}-i_{B1}+N_{A1}-i_{A1}, j_{B2}-i_{B2})
\end{aligned}
$$

Case (viii): $j_{A1} = N_{A1}, j_{A2} < N_{A2}, j_{B1} < N_{B1}, j_{B2} = N_{B2}$. In this case, either skill 1 at B might have been assigned to A, or skill 2 at A might have been assigned to B, in reaching state $\underline{j}$.

$$
\text{(A.8)} \quad
\begin{aligned}
q_{\underline{ij}} = \; & i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}+j_{B1}-i_{B1}, j_{A2}-i_{A2}) + \\
& I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, N_{B2}-i_{B2}+j_{A2}-i_{A2})
\end{aligned}
$$

Case (ix): $j_{A1} < N_{A1}, j_{A2} = N_{A2}, j_{B1} = N_{B1}, j_{B2} < N_{B2}$. In this case, either skill 2 at B might have been assigned to A, or skill 1 at A might have been assigned to B, in reaching state $\underline{j}$.

$$
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\
& I(\underline{i}\in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i}\in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{A2}(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, N_{A2}-i_{A2}+j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{B1}(\underline{j}))\lambda_B p(N_{B1}-i_{B1}+j_{A1}-i_{A1}, j_{B2}-i_{B2})
\end{aligned} \tag{A.9}
$$

Case (x): $j_{A1} < N_{A1}, j_{A2} = N_{A2}, j_{B1} < N_{B1}, j_{B2} = N_{B2}$. In this case, either skill 2 at B might have been assigned to A or skill 2 at A might have been assigned to B in reaching state $\underline{j}$.

$$
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + \\
& I(\underline{i}\in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i}\in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{A2}(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, N_{A2}-i_{A2}+N_{B2}-i_{B2}) \\
& I(\underline{i}\in L_{B2}(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, N_{B2}-i_{B2}+N_{A2}-i_{A2})
\end{aligned} \tag{A.10}
$$

Case (xi): $j_{A1} < N_{A1}, j_{A2} < N_{A2}, j_{B1} = N_{B1}, j_{B2} = N_{B2}$. In this case, either one of or both skill 1 and 2 at A might have been assigned to B in reaching state $\underline{j}$.

$$
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + \\
& I(\underline{i}\in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i}\in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{B1}(\underline{j}))\lambda_B p(N_{B1}-i_{B1}+j_{A1}-i_{A1}, N_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{B2}(\underline{j}))\lambda_B p(N_{B1}-i_{B1}, N_{B2}-i_{B2}+j_{A2}-i_{A2}) + \\
& I(\underline{i}\in L_{B1B2}(\underline{j}))\lambda_B p(N_{B1}-i_{B1}+j_{A1}-i_{A1}, N_{B2}-i_{B2}+j_{A2}-i_{A2})
\end{aligned} \tag{A.11}
$$

Case (xii): $j_{A1} = N_{A1}, j_{A2} = N_{A2}, j_{B1} = N_{B1}, j_{B2} < N_{B2}$. In this case, either one of or both skill 1 and 2 at B might have been assigned to A, or skill 1 at A might have been assigned to B, in reaching state $\underline{j}$.

$$
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\
& I(\underline{i}\in L_A(\underline{j}))\lambda_A p(j_{A1}-i_{A1}, j_{A2}-i_{A2}) + I(\underline{i}\in L_B(\underline{j}))\lambda_B p(j_{B1}-i_{B1}, j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{A1}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}+j_{B1}-i_{B1}, N_{A2}-i_{A2}) + \\
& I(\underline{i}\in L_{A2}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}, N_{A2}-i_{A2}+j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{A1A2}(\underline{j}))\lambda_A p(N_{A1}-i_{A1}+j_{B1}-i_{B1}, N_{A2}-i_{A2}+j_{B2}-i_{B2}) + \\
& I(\underline{i}\in L_{B1}(\underline{j}))\lambda_B p(N_{B1}-i_{B1}+N_{A1}-i_{A1}, j_{B2}-i_{B2})
\end{aligned} \tag{A.12}
$$

Case (xiii): $j_{A1} = N_{A1}, j_{A2} = N_{A2}, j_{B1} < N_{B1}, j_{B2} = N_{B2}$. In this case, either one of or both skill 1 and 2 at B might have been assigned to A, or skill 2 at A might have been assigned to B, in reaching state $\underline{j}$.

$$
\text{(A.13)} \quad
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, N_{A2} - i_{A2}) + \\
& I(\underline{i} \in L_{A2}(\underline{j}))\lambda_A p(N_{A1} - i_{A1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{A1A2}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, N_{B2} - i_{B2} + N_{A2} - i_{A2})
\end{aligned}
$$

Case (xiv): $j_{A1} = N_{A1}, j_{A2} < N_{A2}, j_{B1} = N_{B1}, j_{B2} = N_{B2}$. In this case, either one of or both skill 1 and 2 at A might have been assigned to B, or skill 1 at B might have been assigned to A, in reaching state $\underline{j}$.

$$
\text{(A.14)} \quad
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + N_{B1} - i_{B1}, j_{A2} - i_{A2}) + \\
& I(\underline{i} \in L_{B1}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + N_{A1} - i_{A1}, N_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1}, N_{B2} - i_{B2} + j_{A2} - i_{A2}) + \\
& I(\underline{i} \in L_{B1B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + N_{A1} - i_{A1}, N_{B2} - i_{B2} + j_{A2} - i_{A2})
\end{aligned}
$$

Case (xv): $j_{A1} < N_{A1}, j_{A2} = N_{A2}, j_{B1} = N_{B1}, j_{B2} = N_{B2}$. In this case, either one of or both skill 1 and 2 at A might have been assigned to B, or skill 2 at B might have been assigned to A, in reaching state $\underline{j}$.

$$
\text{(A.15)} \quad
\begin{aligned}
q_{\underline{ij}} \quad = \quad & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + \\
& I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{A2}(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, N_{A2} - i_{A2} + N_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{B1}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, N_{B2} - i_{B2}) + \\
& I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1}, N_{B2} - i_{B2} + N_{A2} - i_{A2}) + \\
& I(\underline{i} \in L_{B1B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, N_{B2} - i_{B2} + N_{A2} - i_{A2})
\end{aligned}
$$

Case (xvi): $j_{A1} = N_{A1}, j_{A2} = N_{A2}, j_{B1} = N_{B1}, j_{B2} = N_{B2}$. In this case, either one of or both skill 1 and 2 at B might have been assigned to A, or either one of or both skill 1 and 2 at A might have been assigned to B, in reaching state $\underline{j}$.

(A.16)

$$\begin{aligned} q_{\underline{ij}} = {} & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + N_{B1} - i_{B1}, N_{A2} - i_{A2}) + \\ & I(\underline{i} \in L_{A2}(\underline{j}))\lambda_A p(N_{A1} - i_{A1}, N_{A2} - i_{A2} + N_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{A1A2}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + N_{B1} - i_{B1}, N_{A2} - i_{A2} + N_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{B1}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + N_{A1} - i_{A1}, N_{B2} - i_{B2}) + \\ & I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1}, N_{B2} - i_{B2} + N_{A2} - i_{A2}) + \\ & I(\underline{i} \in L_{B1B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + N_{A1} - i_{A1}, N_{B2} - i_{B2} + N_{A2} - i_{A2}) \end{aligned}$$

We can consolidate (A.1) – (A.16) by the following, for $\underline{i} \neq \underline{j}$:

$$\begin{aligned} q_{\underline{ij}} = {} & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\ & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\ & I(j_{A1} = N_{A1}) I(\underline{i} \in L_{A1}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, j_{A2} - i_{A2}) + \\ & I(j_{A2} = N_{A2}) I(\underline{i} \in L_{A2}(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) + \\ & I(j_{A1} = N_{A1} \wedge j_{A2} = N_{A2}) I(\underline{i} \in L_{A1A2}(\underline{j}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) + \\ & I(j_{B1} = N_{B1}) I(\underline{i} \in L_{B1}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, j_{B2} - i_{B2}) + \\ & I(j_{B2} = N_{B2}) I(\underline{i} \in L_{B2}(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, N_{B2} - i_{B2} + j_{A2} - i_{A2}) + \\ & I(j_{B1} = N_{B1} \wedge j_{B2} = N_{B2}) I(\underline{i} \in L_{B1B2}(\underline{j}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, N_{B2} - i_{B2} + j_{A2} - i_{A2}) \end{aligned}$$

(A.17)

$$\begin{aligned} = {} & i_{A1}\mu_1 I_{A1}(\underline{i},\underline{j}) + i_{A2}\mu_2 I_{A2}(\underline{i},\underline{j}) + i_{B1}\mu_1 I_{B1}(\underline{i},\underline{j}) + i_{B2}\mu_2 I_{B2}(\underline{i},\underline{j}) + \\ & I(\underline{i} \in L_A(\underline{j}))\lambda_A p(j_{A1} - i_{A1}, j_{A2} - i_{A2}) + I(\underline{i} \in L_B(\underline{j}))\lambda_B p(j_{B1} - i_{B1}, j_{B2} - i_{B2}) + \\ & I(j_{A1} = N_{A1}) I(\underline{i} \in L_{A1}(N_{A1}, j_{A2,}, j_{B1}, j_{B2}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, j_{A2} - i_{A2}) + \\ & I(j_{A2} = N_{A2}) I(\underline{i} \in L_{A2}(j_{A1}, N_{A2,}, j_{B1}, j_{B2}))\lambda_A p(j_{A1} - i_{A1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) + \\ & I(j_{A1} = N_{A1} \wedge j_{A2} = N_{A2}) I(\underline{i} \in L_{A1A2}(N_{A1}, N_{A2,}, j_{B1}, j_{B2}))\lambda_A p(N_{A1} - i_{A1} + j_{B1} - i_{B1}, N_{A2} - i_{A2} + j_{B2} - i_{B2}) + \\ & I(j_{B1} = N_{B1}) I(\underline{i} \in L_{B1}(j_{A1}, j_{A2,}, N_{B1}, j_{B2}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, j_{B2} - i_{B2}) + \\ & I(j_{B2} = N_{B2}) I(\underline{i} \in L_{B2}(j_{A1}, j_{A2,}, j_{B1}, N_{B2}))\lambda_B p(j_{B1} - i_{B1}, N_{B2} - i_{B2} + j_{A2} - i_{A2}) + \\ & I(j_{B1} = N_{B1} \wedge j_{B2} = N_{B2}) I(\underline{i} \in L_{B1B2}(j_{A1}, j_{A2,}, N_{B1}, N_{B2}))\lambda_B p(N_{B1} - i_{B1} + j_{A1} - i_{A1}, N_{B2} - i_{B2} + j_{A2} - i_{A2}) \end{aligned}$$

As usual,

$$q_{\underline{jj}} = -\sum_{\underline{k} \neq \underline{j}} q_{\underline{jk}}$$

This means summing the right side of (A.17). It will be convenient for performing the sum of the indicator functions on the right side of (A.17) to define the following sets which are the related to the sets L_A, L_B, L_{A1}, L_{A2}, L_{B1}, L_{B2}, L_{A1A2}, L_{B1B2} defined above.

$$U_A(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 \geq n_{A1}, a_2 \geq n_{A2}, b_1 = n_{B1}, b_2 = n_{B2}\}$$

$$U_B(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 = n_{A1}, a_2 = n_{A2}, b_1 \geq n_{B1}, b_2 \geq n_{B2}\}$$

$$U_{A1}(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 \geq n_{A1}, a_2 \geq n_{A2}, b_1 > n_{B1}, b_2 = n_{B2}\}$$

$$U_{A2}(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 \geq n_{A1}, a_2 \geq n_{A2}, b_1 = n_{B1}, b_2 > n_{B2}\}$$

$$U_{B1}(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 > n_{A1}, a_2 = n_{A2}, b_1 \geq n_{B1}, b_2 \geq n_{B2}\}$$

$$U_{B2}(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 = n_{A1}, a_2 > n_{A2}, b_1 \geq n_{B1}, b_2 \geq n_{B2}\}$$

$$U_{A1A2}(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 \geq n_{A1}, a_2 \geq n_{A2}, b_1 > n_{B1}, b_2 > n_{B2}\}$$

$$U_{B1B2}(n_{A1}, n_{A2}, n_{B1}, n_{B2}) = \{(a_1, a_2, b_1, b_2) \mid a_1 > n_{A1}, a_2 > n_{A2}, b_1 \geq n_{B1}, b_2 \geq n_{B2}\}$$

Then,

$$q_{\underline{jk}} =$$

$$\begin{gathered}
j_{A1}\mu_1 I_{A1}(\underline{j}, \underline{k}) + j_{A2}\mu_2 I_{A2}(\underline{j}, \underline{k}) + j_{B1}\mu_1 I_{B1}(\underline{j}, \underline{k}) + j_{B2}\mu_2 I_{B2}(\underline{j}, \underline{k}) + \\
I(\underline{j} \in L_A(\underline{k}))\lambda_A p(k_{A1} - j_{A1}, k_{A2} - j_{A2}) + I(\underline{j} \in L_B(\underline{k}))\lambda_B p(k_{B1} - j_{B1}, k_{B2} - j_{B2}) + \\
I(k_{A1} = N_{A1}) I(\underline{j} \in L_{A1}(N_{A1}, k_{A2,}, k_{B1}, k_{B2}))\lambda_A p(N_{A1} - j_{A1} + k_{B1} - j_{B1}, k_{A2} - j_{A2}) + \\
I(k_{A2} = N_{A2}) I(\underline{j} \in L_{A2}(k_{A1}, N_{A2,}, k_{B1}, k_{B2}))\lambda_A p(k_{A1} - j_{A1}, N_{A2} - j_{A2} + k_{B2} - j_{B2}) + \\
I(k_{A1} = N_{A1} \wedge k_{A2} = N_{A2}) I(\underline{j} \in L_{A1A2}(N_{A1}, N_{A2,}, k_{B1}, k_{B2}))\lambda_A p(N_{A1} - j_{A1} + k_{B1} - j_{B1}, N_{A2} - j_{A2} + k_{B2} - j_{B2}) + \\
I(k_{B1} = N_{B1}) I(\underline{j} \in L_{B1}(k_{A1}, k_{A2,}, N_{B1}, k_{B2}))\lambda_B p(N_{B1} - j_{B1} + k_{A1} - j_{A1}, k_{B2} - j_{B2}) + \\
I(k_{B2} = N_{B2}) I(\underline{j} \in L_{B2}(k_{A1}, k_{A2,}, k_{B1}, N_{B2}))\lambda_B p(k_{B1} - j_{B1}, N_{B2} - j_{B2} + k_{A2} - j_{A2}) + \\
I(k_{B1} = N_{B1} \wedge k_{B2} = N_{B2}) I(\underline{j} \in L_{B1B2}(k_{A1}, k_{A2,}, N_{B1}, N_{B2}))\lambda_B p(N_{B1} - j_{B1} + k_{A1} - j_{A1}, N_{B2} - j_{B2} + k_{A2} - j_{A2})
\end{gathered}$$

$$q_{\underline{jj}} = -\sum_{\underline{k} \neq \underline{j}} q_{\underline{jk}}$$

$$
\begin{aligned}
(\text{A.18}) \quad = \quad & -j_{A1}\mu_1 I(j_{A1}>0) - j_{A2}\mu_2 I(j_{A2}>0) - j_{B1}\mu_1 I(j_{B1}>0) - j_{B2}\mu_2 I(j_{B2}>0) - \\
& \sum_{k \in U_A(\underline{j})} \lambda_A p(k_{A1}-j_{A1}, k_{A2}-j_{A2}) - \sum_{k \in U_B(\underline{j})} \lambda_B p(k_{B1}-j_{B1}, k_{B2}-j_{B2}) - \\
& \sum_{k \in U_{A1}(N_{A1}, j_{A2}, j_{B1}, j_{B2})} \lambda_A p(N_{A1}-j_{A1}+k_{B1}-j_{B1}, k_{A2}-j_{A2}) - \\
& \sum_{k \in U_{A2}(j_{A1}, N_{A2}, j_{B1}, j_{B2})} \lambda_A p(k_{A1}-j_{A1}, N_{A2}-j_{A2}+k_{B2}-j_{B2}) - \\
& \sum_{k \in U_{A1A2}(N_{A1}, N_{A2}, j_{B1}, j_{B2})} \lambda_A p(N_{A1}-j_{A1}+k_{B1}-j_{B1}, N_{A2}-j_{A2}+k_{B2}-j_{B2}) - \\
& \sum_{k \in U_{B1}(j_{A1}, j_{A2}, N_{B1}, j_{B2})} \lambda_B p(N_{B1}-j_{B1}+k_{A1}-j_{A1}, k_{B2}-j_{B2}) - \\
& \sum_{k \in U_{B2}(j_{A1}, j_{A2}, j_{B1}, N_{B2})} \lambda_B p(k_{B1}-j_{B1}, N_{B2}-j_{B2}+k_{A2}-j_{A2}) - \\
& \sum_{k \in U_{B1B2}(j_{A1}, j_{A2}, N_{B1}, N_{B2})} \lambda_B p(N_{B1}-j_{B1}+k_{A1}-j_{A1}, N_{B2}-j_{B2}+k_{A2}-j_{A2})
\end{aligned}
$$