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Research Report

Study of Internet Traffic -2, Micro Model of Traffic

Akio Koide

IBM Research, Tokyo Research Laboratory
IBM Japan, Ltd.
1623-14 Shimotsuruma, Yamato
Kanagawa 242-8502, Japan

Research Division
Almaden - Austin - Beijing - Haifa - India - T. J. Watson - Tokyo - Zurich

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1 Introduction

Real Internet traffic is far from the Poisson distribution and does not satisfy the memoryless assumption as I showed in the accompanied paper "The Study of Internet Traffic -1, Extended Poisson Distribution" [1]. The purpose of this report is to develop a mathematical model for generating a correlated traffic. Here we model the traffic as a time sequence just like access log. The time sequence is a mapping from the set of integers (or positive integers) into a set of real numbers. We call a probabilistic algorithm for generating this time sequence a micro model of traffic. We call each element of time sequence a time event .

The simplest micro model is given as follows.

> **Simple Micro Model:**
> The algorithm generates the time event t_{n+1} from t_n by $t_{n+1} = t_n + s_n$ where s_n is the random variable which follows the probability density $p_n(s_n)$.

This is a good starting point to study micro models of traffic but it is not sufficiently correlated to reproduce real Internet traffic as I will show in another accompanied report [2]. The strongly correlated micro model can be given as follows.

> **Compound Micro Model:**
> The algorithm generates t_{n+1} from t_n by $t_{n+1} = t_n + s_n$ where s_n is the random variable which follows to the probability density $p_n(s_n)$. Each t_n probabilistically yields finite length of time sequence $\{t_{n,0}, t_{n,1}, t_{n,2}, ..., t_{n,m}\}$ by sub micro model. Here $t_{n,0}$ represents t_n. The traffic is given by a collection of these fragmented time sequences.

Obviously, there are many algorithms to generate this fragmented time sequence. The simplest one can be given as follows.

> **An Example of Sub Micro Models**
> The algorithm generates the time $t_{n,k+1}$ from $t_{n,k}$ with the probability $P_{n,k}$ by $t_{n,k+1} = t_{n,k} + s_{n,k}$ where $s_{n,k}$ is the random variable which follows to the probability density $p_{n,k}(s_{n,k})$.

By changing the sub model for generating fragmented sequences from one to another, we can easily make many different type of compound micro models. A good micro model of the traffic is simple and sufficiently flexible to reproduce a variety of real Internet traffic with accountable parameters. Therefore, the analysis of the real traffic data is essential. However, in this report, I focus on mathematical formulation for micro model and describes some examples of micro models. I will give the analysis of the real traffic data based on micro model in another report [2].

The observed traffic data are usually statistics of request counts. I will clarify the relation of micro models and statistics of traffic, access counts in time interval. In actual simulation of traffic, we need the starting point of time sequence and face to the problem of stationary state. Firstly I will study the problem of stationary time sequence. Then I propose algorithms for stationary micro model and dynamic micro model for daily/weekly and event-driven behavior of traffic.

2 Simple Micro Model

2.1 Basic Formulation

Let us think about a time sequence $\{t_n\}$ without the start nor end which is generated successively from the random variable s_n by

$$t_{n+1} = t_n + s_n. \tag{1}$$

The index n runs over all integers from $n = -\infty$ to $+\infty$. Let $p(s)$ is the probability density that the random variable s_n takes the value s. We assume the probabilistic causality

$$\begin{aligned} p(s) &\geq 0 \quad \text{if } s \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{2}$$

and the normalization

$$\int_0^\infty p(s)ds = 1. \tag{3}$$

Since the time sequence has no start nor end, there exists some appropriate n for any given time t such that

$$t_n \leq t \leq t_{n+1}. \tag{4}$$

We call t_n the latest time event before the reference time t. Let $\phi(s;t)$ represent the probability density that the latest time event before the reference time t takes the value $t-s$. It satisfies

$$\begin{aligned} \phi(s;t) &\geq 0 \quad \text{if } s \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{5}$$

and

$$\int_0^\infty \phi(s;t)ds = 1. \tag{6}$$

Once the probability density $\phi(s;t)$ is given for some reference time t, we can compute $\phi(s;t')$ for any t' after the time t as follows.

Let us denote $u = t' - t > 0$. When $s > u$, it means that no time event takes place after the latest time event $t_n = t + u - s$ before the next reference $t' = t + u$. Therefore we have

$$\phi(s; t+u) = \phi(s-u;t) \int_s^\infty p(s_1)ds_1 \Big/ \int_{s-u}^\infty p(x)dx \tag{7}$$

for any $s \leq u \leq 0$. When $s \leq u$, the probability density $\phi(s, t+u)$ can be decomposed into the sum of the probability densities that m time events happen after the time $t_n = t - s'$ before the time t'. We have

$$\int\int ... \int \phi(s';t)p(s_1)p(s_2)...p(s_m) \left(\int_s^\infty p(s_{m+1})ds_{m+1} \Big/ \int_{s'}^\infty p(x)dx \right) ds_1 ds_2 ... ds_m \tag{8}$$

where $s_j = t_{n+j} - t_{n+j-1}$ and $s_1 \geq s'$ and $s' = s_1 + s_2 + ... + s_m - (u - s)$.

The result can be expressed in a simple form by using the alternative notations

$$\psi(s;t) = \phi(s;t) \Big/ \int_s^\infty p(x)dx \ , \tag{9}$$

$$\begin{aligned} h_1(s';s'') &= p(s'+s'') \quad \text{if } s' \geq 0 \text{ and } s'' \geq 0, \\ &= 0 \qquad\qquad\quad \text{otherwise,} \end{aligned} \tag{10}$$

and

$$h_m(s';s'') = \int ... \int_{s_1+s_2+...+s_m=s''} h_1(s';s_1)p(s_2)...p(s_m)ds_2...ds_m \tag{11}$$

and

$$h(s';s'') = \sum_{m=1}^\infty h_m(s';s''). \tag{12}$$

Then we have

$$\psi(s;t+u) = \psi(s-u;t) \quad \text{for } s > u \tag{13}$$

and

$$\psi(s;t+u) = \int_0^\infty dx \ \psi(x;t)h(x;u-s) \quad \text{for } s \leq u. \tag{14}$$

By using the same notation, the probability that m time events take place during t and $t+u$ is given by

$$P(0;t,t+u) = \int_0^\infty ds \int_{s+u}^\infty dx \ \psi(s;t)p(x) \tag{15}$$

and

$$P(m;t,t+u) = \int_0^u ds \int_0^\infty dx \int_s^\infty dy \ \psi(x;t)h_m(x;u-s)p(y). \tag{16}$$

2.2 Stationary State

2.2.1 Solution of Stationary State

The stationary state of time sequences is defined as the state that $\phi(s;t)$ is a function of only s and independent from the reference time t. Thus, we can write it as $\phi(s;t) = \phi(s)$. If such a function exists, the corresponding $\psi(s;t) = \psi(s)$ must satisfy both Eq. (13) and Eq.(14).

From Eq. (13), $\psi(s)$ should be constant. By substituting the constant C in $\psi(s;t)$ of Eq.(14), we have

$$\int_0^\infty ds' \ h(s';s'') = 1 \tag{17}$$

in order that $\psi(s)$ can be constant.

Reversely, we can prove Eq.(17) as follows. As shown in the appendix A-1,

$$\int_0^\infty ds' \; h_m(s';s'') = H_{m-1}(s'') - H_m(s'') \tag{18}$$

where

$$H_0(s) = 1 \tag{19}$$

and

$$H_m(s) = \int ... \int_{s_1+...+s_m \le s''} p(s_1)...p(s_m) ds_1...ds_m. \tag{20}$$

Then we have

$$\int_0^\infty ds' \; \sum_{m=1}^{n} h_m(s';s'') = 1 - H_n(s''). \tag{21}$$

As shown in the appendix A-2,

$$H_n(s'') \to 0 \quad \text{when} \;\; n \to \infty \tag{22}$$

as far as some $b > 0$ exists such that

$$\int_0^b p(s) ds < 1. \tag{23}$$

Thus we have

$$\int_0^\infty ds' \; h(s';s'') = 1 \tag{24}$$

The proof ends.

Thus, the constant function

$$\psi(s;t) = C \tag{25}$$

satisfies Eq. (13) and Eq.(14). The constant C is determined by Eq.(5) as

$$1/C = \int_0^\infty ds \int_s^\infty dx \; p(x) = \int_0^\infty dx \; x \, p(x). \tag{26}$$

Then, the probability density for the latest time event is given by

$$\phi(s) = \int_x^\infty dx \; p(x) \Big/ \int_0^\infty dx \; x \, p(x) \tag{27}$$

for the stationary state.

2.2.2 Access Counts in Time Interval

In the stationary state of time sequences, the probability $P(n;t,t+u)$ is also independent from the time t; it can be written as

$$P(n;t,t+u) = P(n;u). \tag{28}$$

The $P(n;u)$ denotes the probability that n time events take place during the interval of the time length u.

By substituting the constant C in Eqs. (15) and (16), we have

$$P(0;u) = C\int_u^\infty ds \int_s^\infty dx\ p(x) = \int_0^\infty ds\ s\,p(s+u) \bigg/ \int_0^\infty ds\ s\,p(s) \tag{29}$$

and

$$P(n;u) = C\int_0^u ds \int_{u-s}^\infty dx\ (H_{n-1}(s) - H_n(s))\,p(x) \quad \text{for } n > 0. \tag{30}$$

They can be alternatively written by

$$P(0;u) = 1 - C\int_0^u ds\ (H_0(s) - H_1(s)) \tag{31}$$

and

$$P(n;u) = C\int_0^u ds\ (H_{n-1}(s) - 2H_n(s) + H_{n+1}(s)) \quad \text{for } n > 0 \tag{32}$$

as shown in the appendixes A-3 and A-4 respectively.

From Eqs.(31) and (32), we have the normalization

$$\sum_{n=0}^{\infty} P(n;u) = 1. \tag{33}$$

The average of counts $\langle n \rangle$ is calculated as

$$\langle n \rangle = \sum_{n=1}^{\infty} nP(n;u) = C\int_0^u ds\ H_0(s) = Cu.$$

Thus we have

$$\langle n \rangle = u \bigg/ \int_0^\infty sp(s)ds \tag{34}$$

The variance $\langle (n - \langle n \rangle)^2 \rangle$ is calculated as

$$\langle (n - \langle n \rangle)^2 \rangle = \sum_{n=1}^{\infty} n^2 P(n;u) - \left(\sum_{n=1}^{\infty} nP(n;u) \right) = Cu + 2C\int_0^u dx\ H(x) - C^2u^2 \tag{35}$$

where

$$H(x) = \sum_{n=1}^{\infty} H_n(x). \tag{36}$$

When the probability density function $p(x)$ is continuous, there exists some appropriate positive number for any given closed interval $[0, b]$ such that

$$0 \leq H(x) \leq Mx \quad \text{for any} \quad x \in [0, b] \tag{37}$$

as is proved in the appendix A-5. Therefore we have

$$\langle (n - \langle n \rangle)^2 \rangle / \langle n \rangle \to 1 \quad \text{as } u \to 0. \tag{38}$$

From the numerical experience, I expect

$$\langle (n - \langle n \rangle)^2 \rangle / \langle n \rangle \to \langle (s - \langle s \rangle)^2 \rangle / \langle s \rangle^2 \quad \text{as } u \to \infty. \tag{39}$$

Here $\langle s^m \rangle$ is the moment defined by

$$\langle s^m \rangle = \int_0^\infty s^m p(s) ds. \tag{40}$$

Thus, generally, $\langle (n - \langle n \rangle)^2 \rangle / \langle n \rangle$ depends on the interval u for event counts at simple micro models.

2.3 Simple Micro Model Algorithms

Here to "generate s with probability density $p(s)$" means to generate a real number s randomly so that $p(s)$ be the probability density that the generated value is s. We assume the probabilistic causality of Eq.(2) for the probability density. Practical techniques to generate s with various functions of $p(s)$ will be described in another Research Report [3]

To "generate m time events of the value t with the probability c_m" means to generate an integer m randomly so that c_m be the probability that the generated integer is m and to repeat the value t by m times in the time sequence. We assume that they satisfy

$$c_0 = 0 \text{ and } \sum_{n=1}^{\infty} c_n = 1. \tag{41}$$

One of ways to generate such m randomly is given as follows.

1. Generate a real number s randomly so that the generated number s distribute uniformly in the interval $(0, 1)$.

2. Return a number m so that

$$\sum_{n=1}^{m-1} c_n \leq s < \sum_{n=1}^{m} c_n$$

2.3.1 Algorithm for Stationary State

Here, we give an algorithm for generating stationary state of time sequence $\{t_1, t_2, t_3, ...\}$ from the reference time t_0 with a given probability density function $p(x)$. The reference time t_0 stands for the starting point of a simulation but is not included in the time sequence.

Parameters :

The parameters to be specified for the algorithm are the starting (reference) point t_0 and the probability density $p(x)$.

Algorithm :

1. Initial Step

(1.1) Generate s_1 with probability density $p_0(s_1)$. Here the function p_0 is defined by

$$p_0(s) = \int_s^\infty p(x)dx \bigg/ \int_0^\infty xp(x)dx \tag{42}$$

(1.2) Compute t_1 by $t_1 = t_0 + s_1$.

2. Successive Step. Do while $n > 1$ until a given termination condition is satisfied:

(2.1) Generate s_n with probability density $p(s_n)$.

(2.2) Compute t_n by $t_n = t_{n-1} + s_n$

(2.3) Increase n by one.

2.3.2 Algorithm for Stationary State with Bulk

Here I will give a probabilistic algorithm for generating stationary state of time sequence by bulk. The 'bulk' means that the events with the same time appear in a time sequence.

Parameters :

The parameters to be specified for the algorithm are the starting (reference) point t_0, the probability density $p(x)$ and the probability c_n. Here c_n denotes the probability that n time events are generated at the same time.

Algorithm :

1. Initial Step

(1.1) Generate s_1 with probability density $p_0(s_1)$. Here the function p_0 is defined by

$$p_0(s) = \int_s^\infty p(x)dx \bigg/ \int_0^\infty xp(x)dx \tag{43}$$

(1.2) Compute t_1 by $t_1 = t_0 + s_1$.

(1.3) Generate m time events of the value t_1 with the probability c_m.

2. Successive Step. Do while $n > 1$ until a given termination condition is satisfied:

(2.1) Generate s_n with probability density $p(s_n)$.
(2.2) Compute t_n by $t_n = t_{n-1} + s_n$
(2.3) Generate m time events of the value t_1 with the probability c_m.
(2.4) Increase n by one.

2.3.3 Dynamic Micro Model

A word 'nonstationary' is somewhat misleading. Therefore I use 'dynamic' for a micro model which simulates daily/weekly change and event-driven change of traffic. Thus I give the algorithm for generating time sequence in probabilistic way such that the expectation value of event count during a unit time interval varies from time to time. In another word, the probability $P(n; t, t+u)$ varies with t as well as with u. Here $P(n; t_1, t_2)$ be the probability that n events take place from the time t_1 to t_2.

Parameters :

The parameters to be specified for the algorithm are the probability density $p(x)$ and the transformation function $T(x)$. We assume that the function $T(x)$ is strongly increasing for $x > 0$. We give the reference time by $t_0 = T(0)$.

Algorithm :

1. Initial Step

(1.1) Generate s_1 with probability density $p_0(s_1)$. Here the function p_0 is defined by

$$p_0(s) = \int_s^\infty p(x)dx \bigg/ \int_0^\infty xp(x)dx \tag{44}$$

(1.2) Compute x_1 by $x_1 = s_1$.
(1.3) Compute t_1 by $t_1 = T(x_1)$.

2. Successive Step. Do while $n > 1$ until a given termination condition is satisfied:

(2.1) Generate s_n with probability density $p(s_n)$.
(2.2) Compute x_n by $x_n = x_{n-1} + s_n$
(2.3) Compute t_n by $t_n = T(x_n)$.
(2.4) Increase n by one.

The point of this algorithm is that we can easily compute probability of dynamic traffic at a given time by use of Eqs (31) and (32). The probability that n time events take place between the times t_1 and t_2 is given by

$$P(n; t_1, t_2) = P(n; X(t_2) - X(t_1)) \tag{45}$$

where $X(t)$ is the inverse function of $T(x)$ such that $T(X(t)) = t$ and $X(T(x)) = x$. The rate of the events $r(t)$ is given as follows.

$$r(t) = \lim_{h\to 0}(1/h)\sum_{n=1}^{\infty} nP(n; t, t+h) = \lim_{h\to 0}(1/h)\sum_{n=1}^{\infty} nP(n; X(t+h) - X(t)).$$

From Eq.(34), we have

$$r(t) = \lim_{h\to 0} C\frac{X(t+h) - X(t)}{h}.$$

Thus we have

$$r(t) = C\frac{dX}{dt} = C \Bigg/ \frac{dT}{dx} \tag{46}$$

where C is defined by Eq.(26).

When we have the rate data $r(t)$ at first, the dynamic micro model may be used with the following steps.

1. Read the parameters of the rate function $r(x)$.
2. Compute $X(t)$ by the integration
$$X(t) = C\int_{t_0}^{t} r(t)dt. \tag{47}$$
3. Construct the inverse function $T(x)$ from the function $X(t)$.
4. Apply dynamic micro model with $T(x)$ and $p(x)$.

However, frequently, we cannot find out a simple analytical form of the inverse function $T(x)$ from $X(t)$ and need to construct a cheap-to-evaluate but accurately approximated function of $T(x)$. Any direct numerical evaluation method for inversing a function requires some iterative procedure and is expensive. In the appendix-B, I will illustrate how to construct an approximation of the inverse function by discussing of the rate model $r(t) = At\exp(-at) + B$ where A, a and B are arbitrary positive numbers.

2.3.4 Construction of Dynamic Micro Model from Count Data

In most applications to real problems, firstly we have a table form of count data, a set of pairs $\{T_n, R_n\}$, but no rate data. Here R_n is the average count at the time interval between T_n and T_{n+1}. One example of this type of data is given in Figure 1 where R_n are hourly hits of IBM Intranet W3 averaged over business days from March 12 to April 6, 2001.

In this case, we can directly construct $T(x)$ by using cubic spline fit of $T(x)$ without the contraction of $r(t)$ nor $X(t)$. The magic is that we know $X(T_n)$ without construction of the functions $r(t)$ nor $X(t)$ because of Eq (47).

$$X(T_{n+1}) - X(T_n) = C\int_{T_n}^{T_{n+1}} r(t)dt = CR_n \tag{48}$$

Therefore we can calculate $X_n = X(T_n)$ successively by $X_{n+1} = X_n + CR_n$ from $X_0 = 0$. Since $T(x)$ is inverse function of $X(t)$, we have also $T(X_n) = T_n$. Thus, the cubic spline fit is applicable to construct $T(x)$.

Figure 1: Hourly report of averaged hits in business day

T_n (hour)	R_n	$C^2 q_n$ (10^{-12})	r_n (/hour)
00	86453	1.768338	90344.5
01	85889	2.226398	82611.8
02	135446	-9.830433	98565.8
03	181704	-0.734637	170866.7
04	193861	-0.451760	192093.6
05	185590	0.658698	187752.9
06	227135	-0.982930	194339.9
07	414082	-1.305133	278821.2
08	749020	-0.218998	590603.9
09	979403	-0.040101	900113.6
10	1052048	-0.011630	1042781.5
11	984815	0.015232	1030566.2
12	914213	0.015310	942884.5
13	889944	0.005762	894158.6
14	886158	-0.005573	893755.3
15	838901	0.021972	860238.3
16	773792	-0.008698	836208.2
17	529516	0.142058	664251.9
18	329415	0.439307	411695.6
19	219596	0.963151	262126.5
20	163237	1.438356	185286.9
21	134316	1.589352	145354.4
22	116437	1.030715	126016.2
23	96275	3.545444	104887.9

Let us approximate $T(x)$ by a polynomial $f_n(x)$ of the degree three for each the interval $[X_n, X_{n+1}]$ as

$$\begin{aligned} f_n(x) &= 1/(CR_n)\left((X_{n+1}-x)T_n + (x-X_n)T_{n+1}\right. \\ &- \left.(X_{n+1}-x)(x-X_n)\left((CR_n + (X_{n+1}-x))q_n + (CR_n + (x-X_n))q_{n+1}\right)\right) \end{aligned} \tag{49}$$

where q_n are unknown parameters. Then at the boundary points, we have

$$f_n(X_n) = T_n \ , \ f_n(X_{n+1}) = T_{n+1}, \tag{50}$$

$$f'_n(X_n) = \frac{T_{n+1}-T_n}{CR_n} - (2q_n + q_{n+1})CR_n \ , \ f'_n(X_{n+1}) = \frac{T_{n+1}-T_n}{CR_n} + (q_n + 2q_{n+1})CR_n, \tag{51}$$

$$f''_n(X_n) = 6q_n \ \text{ and } \ f''_n(X_{n+1}) = 6q_{n+1}. \tag{52}$$

We assume that the polynomials $f_n(x)$ are connected continuously up to their second derivatives at the boundary points. Then, from Eq(51), we obtain the tridiagonal form of simultaneous linear equations on parameters q_n

$$R_{n-1}q_{n-1} + 2(R_{n-1}+R_n)q_n + R_n q_{n+1} = 1/C^2\left(\frac{T_{n+1}-T_n}{R_n} - \frac{T_n - T_{n-1}}{R_{n-1}}\right). \tag{53}$$

In the periodical condition of N intervals, where all properties at T_N are identical to those at T_0, we have N unknown parameters and N linear equations and can solve them. In the open condition of N intervals, where properties at T_0 and T_N may be different, we have only $N-1$ equations and need to reduce the number of unknown parameters $N-1$ by assuming $q_0 = q_N = 0$. The rate can be computed from Eqs (46) and (51) as

$$r_n = \frac{C^2 R_n}{(T_{n+1}-T_n) - (2q_n + q_{n+1})C^2 R_n^2} \tag{54}$$

Figure 1 gives also a solution of periodical condition in terms of $C^2 q_n$ and r_n/C^2 for input data T_n and R_n.

2.4 Memoryless Process

Let us think about a simple micro model given by the probability density

$$\begin{aligned} p(s) &\geq \lambda\exp(-\lambda s) && \text{if } s \geq 0 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{55}$$

This model yields memoryless process in the sense that the probability density of next event time is independent from the last event time;

$$p(s) = p(s'+s) \Big/ \int_{s'}^{\infty} dx \ p(x) \tag{56}$$

Here we will evaluate the functions defined in previous sections by using this probability density.

Firstly we have

$$h_m(s'; s'') = \frac{(\lambda s'')^{m-1}}{(m-1)!} \lambda \exp(-\lambda(s' + s'')). \tag{57}$$

Therefore,

$$h(s'; s'') = \lambda \exp(-\lambda s'). \tag{58}$$

This gives how a state $phi(s, t)$ approaches to the stationary state

$$\phi(s; t+u) = \begin{cases} \exp(-\lambda u)\phi(s-u; t) & \text{for } s > u, \\ \lambda \exp(-\lambda s) & \text{for } s \leq u \end{cases}. \tag{59}$$

The constant C of the stationary state is $C = \lambda$ and the stationary state is

$$\phi(s) = \lambda \exp(-\lambda s). \tag{60}$$

$$H_n(s) = 1 - \exp(-\lambda s) \sum_{m=0}^{n-1} \frac{(\lambda s)^m}{m!} \tag{61}$$

$$H_{n-1}(s) - H_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \exp(-\lambda s). \tag{62}$$

Therefore we have the well-known Poisson distribution [4]

$$P(n; u) = \frac{(\lambda u)^n}{n!} \exp(-\lambda u). \tag{63}$$

The sum of $H_n(x)$ is given by

$$H(x) = \sum_{n=1}^{\infty} H_n(x) = \lambda x. \tag{64}$$

2.4.1 Stationary State with Bulk

Let $B(n; u)$ be the probability that n events takes place during the interval of the time length u in the algorithm for the stationary state with bulk. We will compute $B(n; u)$ from $P(n; u) = (\lambda u)^n / n! \exp(-\lambda u)$, which is the probability that n distinguished time events take place during the interval of the time length u.

Let $W(z; t)$ be a generating function given by

$$W(z; u) = \sum_{n=0}^{\infty} z^n B(n; u). \tag{65}$$

At each distinguished time t_k, m time events can take place with the probability c_m. Therefore, we obtain

$$W(z; u) = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} c_m z^m \right)^n P(n; u). \tag{66}$$

This can be rewritten by

$$W(z;u) = \exp\left(-\lambda(1 - \sum_{m=1}^{\infty} c_m z^m)u\right) \tag{67}$$

This is a generating function of the extended Poisson distribution given in the previous Research Report [1].

3 Compound Micro Model

The compound micro model is a kind of extension of 'bulk' algorithm. In the Algorithm for Stationary State with Bulk of the previous section, m events take place with the probability c_m at the same t_n. In the compound micro model, the m events are generated in some range of separated times by the sub micro model.

The compound micro model can be described as follows.

1. Generate time sequence $\{t_1, t_2, t_3, ...\}$ by some simple micro model.
2. Generate the fragment of time sequence $\{t_{n,0}, t_{n,1} ... t_n, m\}$ by sub micro model for each t_n. We call t_n the parent event of the events of the events $T_{n,k}$.
3. Combine the fragments into one time sequence.
4. Sort the time events $t_{n,k}$ in the combined time sequence in the order of icreasing time.

In the first step, we may use the dynamic simple model to reproduce time-dependent rate of traffic such as daily/weekly change and event-driven change of traffic.

The last step 'sort' is required when we apply a compound micro model for event simulation of networked servers. Therefore a compound micro model is generally more expensive than a simple micro model.

Nevertheless, the compound model has the following merits.

- It can generate a strongly correlated time sequence with a simple probability density function.
- It is relatively accountable in comparison with a simple micro model with a complex density function $p(x)$.

I strongly recommend to use the compound micro model.

In another Research Report [2], I will show how the compound micro model produces observed data.

3.1 Sub Micro Model

The sub micro model is the algorithm for generate fragment of the time sequence. Here I mean by 'fragment' that the number of events in a time sequence is always finite. In this section, I list up typical sub micro models.

In the following, to "continue with the probability P_k means also to terminate the generation of the next sub time events with the probability $1 - P_k$.

Any sub micro model can be characterized quantitatively by the average length $\langle m \rangle$ of fragment and expected event distribution density $n(s)$ which denotes expected number of distribution at the relative time s after the parent event. From this definition, we have

$$\langle m \rangle = \int_0^\infty ds n(s). \tag{68}$$

3.1.1 Basic Sub Micro Model

Parameters :

The parameters to be specified in the algorithm are the start time t_n, the probability density function $q(s)$ and the probability P_k.

Algorithm :

1. Initial Step: Set $t_{n,0} = t_n$.
2. Successive Step ($k > 0$): Continue with the probability P_k.
   - (2.1) Generate $s_{n,k}$ with probability density $q(s_{n,k})$.
   - (2.2) Compute $t_{n,k}$ by $t_{n,k} = t_{n,k-1} + s_{n,k}$
   - (2.3) Increase k by one.

The average length of generated sequence is given by

$$< m >= \sum_{m=1}^{\infty} m \prod_{k=1}^{m} P_k. \tag{69}$$

We force P_k that the above summation converges. This condition is obviously satisfied when there exists upper bound $\epsilon < 1$ for sufficiently larger k such that $P_k < \epsilon$.

3.1.2 Modulated Sub Micro Model

Parameters :

The parameters to be specified in the algorithm are the start time t_n, the probability density functions $q(s)$ and $q'(s)$ and the probabilities P_k and P'_k.

Algorithm :

1. Initial Step: Set $t_{n,0} = t_n$.
2. Set $k = k' = 1$.
3. Outer Loop ($k > 0$): Continue with the probability P_k.
   - (3.1) Generate $s_{n,k'}$ with probability density $q(s_{n,k'})$.
   - (3.2) Compute $t_{n,k'}$ by $t_{n,k'} = t_{n,k'-1} + s_{n,k'}$.
   - (3.3) Set $l = 1$.
   - (3.4) Inner Loop ($l > 0$): Continue with the probability P'_l.

i. Generate $s_{n,k'+l}$ with probability density $q'(s_{n,k'+l})$.

ii. Compute $t_{n,k'+l}$ by $t_{n,k'+l} = t_{n,k'+l-1} + s_{n,k'+l}$.

iii. Increase l by one.

(3.5) Increase k by one.

(3.6) Replace k' by $k' + l$.

3.1.3 Compound Sub Micro Model

A compound sub micro model is the sub micro model which calls a sub micro model inside. If it calls itself, it is recurrence type of compound micro model.

Parameters :

The parameters to be specified in the algorithm are the start time t_n, the probability density function $q(s)$, the probability P_k and sub model to be called inside.

Algorithm :

1. Initial Step: Set $t_{n,0} = t_n$.
2. Set $k = 1$.
3. Outer Loop ($k > 0$): Continue with the probability P_k.

(3.1) Generate $s_{n,k}$ with probability density $q(s_{n,k})$.

(3.2) Compute $t_{n,k}$ by $t_{n,k} = t_{n,k-1} + s_{n,k}$.

(3.3) Call a sub model with $t_{n,k}$ and generate a sub fragment of time sequence $\{t_{n,k,0}, t_{n,k,1}, ...t_{n,k,l}\}$.

(3.4) Combine generated sub fragment with previous combined sub fragment.

(3.5) Sort it.

(3.6) Increase k by one.

3.1.4 Stationary Compound Model

In stationary state, the rate r of the compound model is given by

$$\langle n \rangle = \langle m \rangle \Big/ \int_0^\infty xp(x)dx \tag{70}$$

where $p(s)$ is the probability density for the parent events. The expected count $Average$ in the interval of time length u is given by ru.

In any event simulation of compound micro model, the existence of starting point t_0 causes some error because of neglecting time fragments which are generated from the parent events before the starting point t_0 although this error decreases with elapse time s. The expected error $e(s)$ of the rate at the elapse time s is given by

$$e(s) = \int_s^{infty} n(x)dx \Big/ \int_0^\infty xp(x)dx \ . \tag{71}$$

Thus, the relative error is estimated

$$e_r(s) = \int_s^{infty} n(x)dx \Big/ \int_0^{\infty} n(x)dx \ . \tag{72}$$

In the event simulation of compound micro model, we need to compute statistics of stationary after sufficiently large elapse time s where $e_r(s)$ can be negligible with respect to required accuracy.

3.1.5 Dynamic Compound Model

Dynamic compound model can be obtained by applying dynamic micro model algorithm of previous section to generate the parent events t_n. Of course, dynamic compound model may be realized by making the parameters of sub micro model dependent on the parent time. However, the analysis of real data does not require the time dependence of sub micro model parameters. In the report [1], we found the ratio of the variance to the average is almost constant while the average varies largely as daily change.

On the other hand, frustrated users may change their behavior in web access according to the status of end-to-end performance. In this case, I expect the change of the sub micro model parameters will be effective to model user behavior to end-to-end performance.

References

[1] Akio Koide, *Study of Internet traffic -1. Extended Poisson Distribution*, IBM Research Report RT0428, 2001.

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[3] Akio Koide, *Random Variable Generator for Performance Simulation. Analysis of Real Data*, IBM Research Report RT0431, 2001.

[4] Donald Gross and Carl M. Harris, *Fundamentals of Queueing Theory*, Second Edition, John Wiley & Sons, ISBN 0-471-89067-7, 1985.

A Proofs

A.1 Proof of Eq.(18); Integration of $h_m(s'; s'')$

Let us simplify the integral

$$\int_0^\infty ds'\ h_m(s'; s'') = \int_0^\infty \dots \int_{s_1+s_2\dots s_m=s''} p(s'+s_1)p(s_2)\dots p(s_m)ds'ds_2\dots ds_m \tag{73}$$

by defining a new variable of integration s_1 by $s'+s_1$. Then we have

$$\int_0^\infty ds'\ h_m(s'; s'') = \int \dots \int_{C(s'')} p(s_1)p(s_2)\dots p(s_m)ds_1 ds_2\dots ds_m \tag{74}$$

where $C(s'')$ is a region for integration given by

$$s_1 + s_2 + \dots + s_m > s'' \tag{75}$$

and

$$s_2 + \dots + s_m \le s''\ . \tag{76}$$

All the points of the set specified by

$$s_1 + s_2 + \dots + s_m \le s'' \tag{77}$$

satisfies Eq.(76). Therefore, the region $C(s'')$ is given by subtracting the set of Eq.(77) from the set of (76). Thus, Eq.(18) is proved.

A.2 Proof of Eq.(22); Evaluation of $H_n(s)$

Suppose that some $b > 0$ exists such that

$$\int_0^b p(s)ds = \epsilon < 1. \tag{78}$$

Then, as $p(s) \ge 0$, we have

$$\int_0^x p(s)ds \le \epsilon \quad \text{for any } x \le b. \tag{79}$$

Also there exists an appropriate integer k for a given s'' such that

$$(k+1)b > s''. \tag{80}$$

Let us divide the region for the integration of Eq.(20) into $m!/(k!(m-k)!)$ of subregions which k parameters of s_j are larger than any of the remaining parameters. Then we have

$$H_m(s'') = \frac{m!}{k!(m-k)!} \int \dots \int_{R_m} ds_1\dots ds_m\ p(s_1)\dots p(s_m) \tag{81}$$

where the integration region R_m is given by $s_1 + s_2 + ... + s_m \leq s''$ and

$$s_j \geq s_l \quad \text{for any } j,\, k \text{ such that } j \leq k < l. \tag{82}$$

On the other hand, from Eq.(80), the number of parameters s_i beyond b can not exceed k. Therefore $s_l \leq b$ for $l > k$. We have

$$H_m(s'') \leq \frac{m!}{k!(m-k)!} \left(\int_0^\infty ds\, p(s) \right)^k \left(\int_0^b ds\, p(s) \right)^{m-k}. \tag{83}$$

Finally, we have

$$H_m(s'') \leq \frac{m!}{k!(m-k)!} \epsilon^{m-k}. \tag{84}$$

And

$$H_m(s'') \to 0 \tag{85}$$

when n becomes infinite.

A.3 Proof of Eq.(31)

$$\begin{aligned} P(0;u) &= C \int_0^\infty ds \int_s^\infty dx\, p(x) - C \int_0^u ds \int_s^\infty dx\, p(x) \\ &= 1 - C \int_0^u ds\, \left(1 - \int_0^s dx\, p(x) \right) \\ &= 1 - C \int_0^u ds\, (H_0(s) - H_1(s)). \end{aligned} \tag{86}$$

The proof ends.

A.4 Proof of Eq.(32)

$$\begin{aligned} \frac{d}{du} \int_0^u ds \int_{u-s}^\infty dx\, H_n(s) p(x) &= H_n(u) \int_0^\infty dx\, p(x) - \int_0^u ds\, H_n(s) p(u-s) \\ &= H_n(u) - H_{n+1}(u). \end{aligned} \tag{87}$$

Therefore we have

$$\int_0^u ds \int_{u-s}^\infty dx\, H_n(s) p(x) = \int_0^u dx\, (H_n(x) - H_{n+1}(x)). \tag{88}$$

This leads to Eq.(32). The proof ends.

A.5 The Alternative Expression for $H(x)$ and Proof of Eq(37)

By noting that

$$H_{n+1}(x) = \int_0^x dy\ p(x-y)H_n(y), \tag{89}$$

we have

$$H(x) = \int_0^x dy\ p(y) + \int_0^x dy\ p(x-y)H(y) \tag{90}$$

Since each $H_n(x)$ is nonnegative and increasing, the sum $H(x)$ is also nonnegative and increasing.

$$H(y) \leq H(x) \ \text{ if } \ y \leq x.$$

This and Eq.(90) lead to

$$H(x) \leq \int_0^x dy\ p(y) + H(x)\int_0^x dyp(x-y).$$

Then we have

$$\int_0^x dy\ p(y) \leq H(x) \leq \int_0^x dyp(y) \Big/ \left(1 - \int_0^x dy\ p(y)\right)\ . \tag{91}$$

As $p(x)$ is continuous, there is a maximum number L of $p(x)$ for any given closed interval $[0,b]$; there exists L such that $p(x) \leq L$ for any $x \in [0,b]$. This and Eq(91) lead

$$0 \leq H(x) \leq \frac{Lx}{1 - \int_0^b dy\ p(y)}. \tag{92}$$

The proof ends.

A.6 Evaluation of H(x) by Mean Value Theorem

A.6.1 The integral form of mean value theorem

Let extend the mean value theorem into an integral form. Suppose $f(x)$ and $g(x)$ are continuous and $g(x) \neq 0$ in the closed interval $[a,b]$. Let $F(t)$ denote the integral

$$F(t) = \int_a^t f(x)g(x)dx \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \int_a^t g(x)dx. \tag{93}$$

Then we have $F(a) = F(b) = 0$. From the Rolle Theorem, there exists t such that $F'(t) = 0$ and $a < t < b$. As $g(x) \neq 0$, it leads to the integral form of mean value theorem: There exists t such that

$$\int_a^b f(x)g(x)dx = f(t)\int_a^b g(x)dx \quad \text{and} \quad a < t < b. \tag{94}$$