RZ 3100 (# 93146) 01/25/99
Computer Science/Mathematics 5 pages

# Research Report

## On the Redundancy Ratio of Irredundant Sum-of-Product Forms

L. O'Connor

IBM Research Division
Zurich Research Laboratory
8803 Rüschlikon
Switzerland



Research Division
Almaden · Austin · Beijing · Haifa · T.J. Watson · Tokyo · Zurich

# On the Redundancy Ratio of Irredundant Sum-of-Product Forms

L. O'Connor

*IBM Research Division, Zurich Research Laboratory, 8803 Rüschlikon, Switzerland*

## Abstract

The redundancy ratio of a function is defined as the ratio of the length of its longest sum-of-products expression to the length of its shortest sum-of-products expression. For a given function $f$, its redundancy ratio will be denoted by $\rho(f)$. In [8] it was shown that $\rho(f)$ is less than $2^{n-1}$ for boolean functions, and in some cases is a linear function of the number of input variables. In this paper we prove that $\rho(f) \leq 2n$ with high probability for a boolean function of $n$-variables.

# 1 Introduction

As noted [8], many logic minimization algorithms for a boolean function $f$ are based on constructing the set of prime implicants $PI(f)$ for $f$, and then selecting an irredundant sum-of-products (ISOP) expression $p_1+p_2+\cdots+p_k$ for $f$ where $p_i \in PI(f)$. The *size* or *length* of an ISOP is equal to the number of product terms it contains, and logic minimization algorithms endeavor to find minimal or short ISOPs for given functions. In general, selecting the minimal length ISOP is difficult as it is an instance of the NP-hard set covering problem [1]. Sasao and Butler [8] examined the potential discrepancy between the true minimal ISOP for a function and an ISOP returned by a given minimization algorithm, using the following definitions.

**Definition 1.1** The number of product terms in a minimal sum-of-products expression ($MSOP$) for $f$ is denoted as $\tau(MSOP, f)$, while the worst sum-of-products ($WSOP$) is denoted as $\tau(WSOP, f)$. The *redundancy ratio* $\rho(f)$ is defined as

$$\rho(f) \quad = \quad \frac{\tau(WSOP, f)}{\tau(MSOP, f)}. \tag{1}$$

□

A low redundancy ratio $\rho(f)$ implies almost all ISOPs are minimal, while a high ratio indicates that a given minimization algorithm may produce an ISOP that is far from minimal. Sasao and Butler [8] considered a class $ST(n, k)$ of $n$-variable symmetric functions and showed that $\rho(ST(n, k))$ is less than 2, However when $n = mr$, they also showed that this class of functions can be combined to give new functions $ST(m, k)^r$ for which $\rho(ST(m, k)^r) = c^r$ where $c > 1$. Further, they also gave experimental evidence to show that the ISOP generator of Minato [4] yields WSOPs for $ST(m, k)^r$, and also presented several results for multiple-valued functions.

In this correspondence our main result is to prove a general statement concerning the redundancy ratio $\rho(f)$ that holds with *high probability* for boolean functions, and applies to *all* ISOP minimization algorithms. In particular, assuming the uniform distribution on the set of $n$-variable boolean functions, we prove that $\rho(f) \leq 2n$ with probability tending to 1 as an exponential function of $n$. Thus with high probability *any* minimization algorithm will produce ISOPs for an $n$-variable function whose size is no more than $2n$ times larger than minimal ISOPs. We note that this is not an average case analysis of $\rho(f)$. With our results we can for example state that $\rho(f) \leq 2n$ when $n = 64$ with probability approximately $10^{-10}$, and with probability approximately $10^{-25}$ when $n = 128$.

# 2 Bounding the redundancy ratio

Basic definitions of switching theory can be found in [5], but a partial review is presented here. A variable $x$ has *literals* $x$ and $\bar{x}$, and a *product* term is the logical AND of distinct literals. The logical OR of several product terms is called a *sum-of-products* (SOP) expression. For $n$-variable boolean functions, a product term $p = x_1' x_2 \cdots x_k'$ is said to have *length* $|p| = k$, $1 \leq k \leq n$, and a *minterm* is a product term of length $n$. An *implicant* of $f$ is a product term $p$ that implies $f$, and implicants of length $k$ imply or cover $2^{n-k}$ minterms of $f$. Implicants of maximum size are called *prime*, and let $PI(f)$ be the set of *prime implicants* for $f$. For $PI(f)$ we define $\chi(f)$ as

$$\chi(f) \quad = \quad \min_{p \in PI(f)} |p| \tag{2}$$

which indicates that no $p \in PI(f)$ covers more than $2^{n-\chi(f)}$ minterms of $f$. An *irredundant SOP* (ISOP) for $f$ is a SOP expression $f = p_1 + p_2 + \cdots + p_k$, $p_i \in PI(f)$, where removing any $p_j$ results in a function that does not imply $f$. Finally, let the *weight* $w(f)$ of an $n$-variable boolean function $f$ be defined as

$$w(f) = \#\{X \mid f(X) = 1, X \in \{0,1\}^n\}. \tag{3}$$

where $\#S$ denotes the cardinality of the set $S$. The weight of $f$ is equivalent to the number of minterms in $f$. We now define a general bound on $\rho(f)$ in terms of $\chi(f)$. which makes use of the fact that any ISOP for $f$ must have at least $w(f)/2^{n-\chi(f)}$ product terms. We note that this observation has been used previously to lower bound ISOP lengths [?7].

**Lemma 2.1** For an $n$-variable boolean function, $\rho(f) \leq 2^{n-\chi(f)}$.

*Proof.* The lemma is proved by upper bounding $\tau(WSOP, f)$, and lower bounding $\tau(MSOP, f)$. We first observe that for a given function $f$ we can minimise $f$ or its complement $\bar{f}$, so that without loss of generality we can assume that $w(f) \leq 2^{n-1}$. It then follows that $\tau(WSOP, f) \leq w(f)$, and this upper bound is tight since $f$ can always be selected so that $\forall p \in PI(f)$, $|p| = n$. On the other hand, each prime implicant of $p \in PI(f)$ covers at most $2^{n-\chi(f)}$ minterms of $f$, implying that $\tau(MSOP, f) \geq w(f)/2^{n-\chi(f)}$. Combining these bounds we then see that

$$\rho(f) = \frac{\tau(WSOP, f)}{\tau(MSOP, f)} \leq \frac{w(f)}{w(f) \cdot 2^{\chi(f)-n}} \leq 2^{n-\chi(f)}. \tag{4}$$

which proves the lemma. □

An analysis of $\chi(f)$ is then required to apply Lemma 2.1. If an implicant $p$ has length $k$, then the $2^{n-k}$ minterms implied by $p$ correspond to $2^{n-k}$ vertices of the $n$-cube $Q_n$ and form a subgraph known as a $(n-k)$-cube [2]. There is a one-to-one correspondence between implicants of length $k$ and cubes of dimension $(n-k)$ in $Q_n$, and cubes of large dimension correspond to implicants of short length. Thus $\chi(f)$ is the length of the shortest prime implicant of $f$, and thus determines the largest subcube in $Q_n$ implied by a prime implicant of $f$.

**Corollary 2.1** For a uniformly selected $n$-variable boolean function $f$ of weight $0 \leq w(f) \leq 2^n$, $\Pr(\chi(f) = k) \leq G(n, w(f), n-k)$, where $G(n, w(f), n-k)$ is the expected number of $(n-k)$-cubes for a function of weight $w(f)$.

*Proof.* Observe that

$$\begin{aligned} G(n, w(f), n-k) &= \sum_{i \geq 0} i \cdot \Pr(f \text{ has } i \text{ implicants of length } k), \\ &\geq \sum_{i \geq 0} i \cdot \Pr(f \text{ has } i \text{ prime implicants of length } k), \\ &\geq \Pr(\chi(f) = k). \end{aligned}$$

□

We observe that the bound derived in Corollary 2.1 is only useful for those values of $k$ for which $G(n, w(f), n-k) < 1$. As will be shown below $G(n, w(f), \lceil \log n \rceil + 1) \to 0$ as a function of $n$, from which we will prove that $\chi(f) \leq (n - \lceil \log n \rceil)$ with high probability.

**Theorem 2.1** Assuming the uniform distribution on the set of $n$-variable boolean functions such that $w(f) \leq 2^{n-1}$, then as $n \to \infty$,

$$\Pr(\chi(f) > (n - \lceil \log n \rceil)) \;\; < \;\; 2^{\lceil \log^2 n \rceil - \lceil \log n \rceil - n}. \tag{5}$$

where $\lceil \log^2 n \rceil = \lceil \log n \rceil \cdot \lceil \log n \rceil$.

*Proof.* We first observe that $\Pr(\chi(f) = k)$ increases monotonically as the weight of $f$ increases, and bounding this probability for the case where $w(f) = 2^{n-1}$ also provides a bound for the cases where $w(f) \leq 2^{n-1}$. From [3] we have that

$$G(n, w(f), k) \;\; = \;\; \frac{2^{n-k} \cdot \binom{n}{k}}{\binom{2^n}{w(f)}} \cdot \binom{2^n - 2^k}{w(f) - 2^k}. \tag{6}$$

When $w(f) = 2^{n-1}$ this can be simplified to

$$G(n, 2^{n-1}, k) \;\; = \;\; \frac{\binom{n}{k} \cdot 2^{n-k} \cdot \binom{2^n - 2^k}{2^{n-1} - 2^k}}{\binom{2^n}{2^{n-1}}} \;\; = \;\; \binom{n}{k} \cdot 2^{n-k} \cdot \frac{(2^{n-1})^{\underline{2^k}}}{(2^n)^{\underline{2^k}}}. \tag{7}$$

When $k = o(\sqrt{n})$ the quotient in (7) can be approximated as

$$\begin{aligned} \frac{(2^{n-1})^{\underline{2^k}}}{(2^n)^{\underline{2^k}}} \;\; &= \;\; 2^{-2^k} \cdot \exp\left[ \sum_{i=0}^{2^k - 1} \ln\left(1 - \frac{i}{2^{n-1}}\right) - \ln\left(1 - \frac{i}{2^n}\right) \right] \\ &= \;\; 2^{-2^k} \cdot \left[ 1 - \frac{2^k(2^k - 1)}{2^{n-1}} + \frac{2^k(2^k - 1)}{2^n} + O\left(\frac{2^{3k}}{2^{2n-2}}\right) \right]. \end{aligned} \tag{8}$$

Then as $n \to \infty$ with $k = \lceil \log n \rceil + 1$, it follows from eqs. (7) and (8) that

$$G(n, 2^{n-1}, \lceil \log n \rceil + 1) \;\; < \;\; \binom{n}{\lceil \log n \rceil + 1} \cdot \frac{2^{n - \lceil \log n \rceil - 1}}{2^{2^{\lceil \log n \rceil + 1}}} \;\; < \;\; 2^{\lceil \log^2 n \rceil - \lceil \log n \rceil - n} \tag{9}$$

where the binomial coefficient in (9) was bound as $2^{\lceil \log^2 n \rceil}$. We observe that the series $\sum_{k>0} G(n, 2^{n-1}, k)$ is unimodal since

$$\frac{G(n, 2^{n-1}, k+1)}{G(n, 2^{n-1}, k)} \;\; = \;\; \frac{n-k}{2(k+1)} \cdot \frac{(2^{n-1} - 2^k)^{\underline{2^k}}}{(2^n - 2^k)^{\underline{2^k}}}$$

and the first term on the right hand side is unimodal, and the second term is always less than 1. Further, $2 \cdot G(n, 2^{n-1}, \lceil \log n \rceil + 1)$ can be used to bound the tail of the series for terms $k > \lceil \log n \rceil$. Thus as $n \to \infty$

$$\Pr(\chi(f) > (n - \lceil \log n \rceil)) \;\; \leq \;\; \sum_{k > \lceil \log n \rceil} G(n, 2^{n-1}, k) \;\; \leq \;\; 2^{\lceil \log^2 n \rceil - \lceil \log n \rceil - n}.$$

The theorem now follows. □

**Corollary 2.2** Assuming the uniform distribution of the set of $n$-variable boolean functions such that $w(f) \leq 2^{n-1}$, then as $n \to \infty$

$$\Pr(\rho(f) > 2n) \;\; < \;\; 2^{\lceil \log^2 n \rceil - \lceil \log n \rceil - n}. \tag{10}$$

*Proof.* Theorem 2.1 states that $\chi(f) \leq (n - \lceil \log n \rceil)$ with high probability as $n \to \infty$. Then from Lemma 2.1 it follows that $\rho(f) \leq 2^{n - (n - \lceil \log n \rceil)} = 2^{\lceil \log n \rceil} < 2n$ with high probability as $n \to \infty$. □

| $n$ | $\lceil \log n \rceil + 1$ | $G(n, 2^{n-1}, \lceil \log n \rceil + 1)$ | $\sum_{k > \lceil \log n \rceil + 1} G(n, 2^{n-1}, k)$ |
|---|---|---|---|
| 4 | 3 | $0.17404 \times 10^{-1}$ | 0 |
| 5 | 4 | $0.15139 \times 10^{-4}$ | 0 |
| 6 | 4 | $0.13698 \times 10^{-3}$ | $< 10^{-11}$ |
| 7 | 4 | $0.42990 \times 10^{-3}$ | $< 10^{-8}$ |
| 8 | 4 | $0.96589 \times 10^{-3}$ | $< 10^{-7}$ |
| 9 | 5 | $0.18316 \times 10^{-7}$ | $< 10^{-17}$ |
| 10 | 5 | $0.46717 \times 10^{-7}$ | $< 10^{-16}$ |
| 11 | 5 | $0.96159 \times 10^{-7}$ | $< 10^{-15}$ |
| 12 | 5 | $0.17442 \times 10^{-6}$ | $< 10^{-15}$ |
| 13 | 5 | $0.29126 \times 10^{-6}$ | $< 10^{-15}$ |
| 14 | 5 | $0.45972 \times 10^{-6}$ | $< 10^{-14}$ |
| 15 | 5 | $0.69438 \times 10^{-6}$ | $< 10^{-14}$ |
| 16 | 5 | $0.10135 \times 10^{-5}$ | $< 10^{-14}$ |

Table 1: Displaying the bounds derived in Theorem 2.1 for $4 \leq n \leq 12$.

# 3 Conclusion

The bound on the probability $\Pr(\chi(f) > (n - \lceil \log n \rceil)$ of Theorem 2.1 is asymptotic, as is the bound on the probability $\Pr(\rho(f) > 2n)$ given in Corollary 2.2. For small $n$, most accuracy is lost when the binomial coefficient $\binom{n}{k}$ of $G(n, 2^{n-1}, k)$ is bounded as $n^k$ in eq. (9), but more accuracy can be acquired if $G(n, 2^{n-1}, k)$ is evaluated directly. Some numerical values $G(n, 2^{n-1}, \lceil \log n \rceil + 1)$ are shown in Table 1, and we see that even for small $n$ the bound can be used to make reasonable statements. Considering Corollary 2.2 by evaluating the bound on the RHS of (10) we find that $\Pr(\rho(f) > 2n)$ for $n = 32, 48, 64$ and 128 is $0.244 \times 10^{-3}$, $0.3814 \times 10^{-5}$, $0.582 \times 10^{-10}$ and $0.129 \times 10^{-25}$, respectively.

The proof of Theorem 2.1 is based on functions of weight $w(f) = 2^{n-1}$, which gives the general bound of Corollary 2.2. Clearly more accurate bounds on $\chi(f)$ (and hence bounds on $\rho(f)$) can be derived for classes of functions for which $w(f) < 2^{n-1}$. Extending the approach of Theorem 2.1 for functions of weight less than $2^{n-1}$ has been considered in [6].

# References


[1] M. R. Garey and D. S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-completeness.* San Francisco, W. H. Freeman and Co. , 1979.

[2] M. A. Harrison, *Introduction to Switching and Automata Theory.* McGraw-Hill, Inc. , 1965.

[3] F. Mileto and G. Putzolu, "Average values of quantities appearing in boolean function minimization," *IEEE Transactions on Electronic Computers*, EC-13:87–92, 1964.

[4] S. Minato, "Fast generation of prime-irredundant covers from binary decision diagrams," *IEICE Trans. Fundamentals*, E76-A(6):973–976, 1993.

[5] S. Muroga, *Logic design and switching theory.* John Wiley and Sons, 1979.

[6] L. J. O'Connor, "A new lower bound on the expected size of irredundant forms for boolean functions," *Information Processing Letters*, 53:347–353, 1995.

[7] T. Sasao, "Bounds on the average number of product terms in the minimum sum-of-products expressions for multiple-valued input two-valued output functions," *IEEE Transactions on Computers*, 40(5):645–651, 1991.

[8] T. Sasao and J. Butler, "Comparison of worst and best sum-of-product expressions for multiple-valued functions," In *IEEE International Symposium on Multiple-Valued Logic, Nova Scotia, Canada, May 28–30*, pages 55–60, 1997.