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Research Report

Rate-$(N-1)/N$ Convolutional Codes with Optimal Spectrum

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Abstract

New recursive, systematic rate-$(N-1)/N$ convolutional encoders for $2 \leq N \leq 20$ and memory $1 \leq M \leq 6$ are presented. These encoders generate codes having optimal distance spectra and were obtained by performing an efficient search. The search possibilities were limited by exploiting the struture of the encoder using various combinatorial arguments. Many of the codes improve upon previously reported results and are attractive for use in high data-rate applications in conjunction with iterative decoding schemes.

I. Introduction

High-rate error control codes are desirable for communications and data storage applications requiring very high data rates. For example, data rates in today's magnetic hard-disk drives exceed 1 Gb/s. The need for high-rate codes is more critical in magnetic recording systems because the channel quality deteriorates as a quadratic function of the rate. Furthermore, most high-performance storage and communications systems employ a concatenated coding scheme in which the component codes must have rates higher than the overall system code rate. Concatenated coding and iterative decoding schemes are being investigated for application to magnetic recording. Recently, the use of rate-$(N-1)/N$ tail-biting codes in a magnetic recording system employing iterative detection/decoding was investigated [1]. The use of soft-in/soft-out decoding based on the rate-$1/N$ dual code allows significant reduction in complexity and alleviates the need for using the traditional approach of puncturing to obtain high-rate codes.

In this report, we present the results of our search for rate-$(N-1)/N$ convolutional codes. Our goal has been to find codes with $2 \leq N \leq 20$ and memory $1 \leq M \leq 6$. Our criterion for selecting good codes is the distance spectrum. Here, we report results based on selecting the best first eight spectral coefficients. Significant reduction in the number of seach possibilities was obtained by employing combinatorial arguments described in the sequel. We report codes with improved spectrum compared with known codes at similar rates reported in the literature [2], [3], [4], [5], [6]. Note that we have followed the popular practice of presenting and describing the properties of a convolutional code when, in fact, they are mostly the properties of the encoder that generates the code.

Let $\mathbf{x}_t = [x_{t,1}\, x_{t,2} \ldots x_{t,K}]$, $t = 0, 1, \ldots,$ $x_{t,i} \in \mathbb{F}_2$, $\mathbb{F}_2 = \{0, 1\}$, be the sequence of information vectors denoted by $\mathbf{x}(D) = \mathbf{x}_0 + \mathbf{x}_1 D + \mathbf{x}_2 D^2 + \ldots$ and let $\mathbf{y}_t = [y_{t,1}\, y_{t,2} \ldots y_{t,N}]$, $y_{t,i} \in \mathbb{F}_2$, be the sequence of code vectors denoted by $\mathbf{y}(D) = \mathbf{y}_0 + \mathbf{y}_1 D + \mathbf{y}_2 D^2 + \ldots$. Considered are rate $R = (N-1)/N$ convolutional codes, i.e., $K = N-1$, which can be encoded using the $(N-1) \times N$ matrix

$$\mathbf{G}(D) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & g_1(D)/g_0(D) \\ 0 & 1 & 0 & \ldots & 0 & g_2(D)/g_0(D) \\ & & \ddots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & g_{N-1}(D)/g_0(D) \end{bmatrix},$$

where $g_i(D) = g_{i,0} + g_{i,1} D + \ldots + g_{i,M} D^M$, $g_{i,j} \in \mathbb{F}_2$, are polynomials of maximum degree M:

$$M = \max_{i=0,1,\ldots,N-1} \deg(g_i(D)).$$

We will call the $g_i(D)$ generators and $g_0(D)$ in particular the recursive part, which must be non-trivial, i.e., $g_0(D) \neq 0$. Let the set $\mathcal{C}$ contain all valid code sequences $\mathbf{y}(D) = \mathbf{x}(D)\mathbf{G}(D)$ encoded from $\mathbf{x}(D)$. Any $\mathbf{y}(D)$ must fulfill the parity check equation $\mathbf{y}(D)\mathbf{H}(D)^T = \mathbf{0}(D)$, where

$$\mathbf{H}(D) = [g_1(D)\, g_2(D) \ldots g_{N-1}(D)\, g_0(D)\,]$$

is called the parity check matrix or the syndrome former of the code. The code is completely specified by the N generators in $\mathbf{H}(D)$ and we therefore favor to address a code by its syndrome former rather than $\mathbf{G}(D)$.

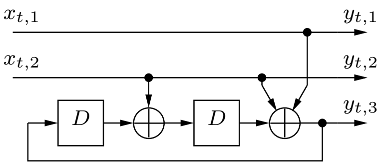

Fig. 1. Encoder of the rate-2/3 code $\mathbf{H}(D) = [1\ 1{+}D\ 1{+}D^2]$.

The encoder $\mathbf{G}(D)$ of the code is minimal and basic if it is realized in observer canonical form [2], e.g., Figure 1 depicts the encoder of the rate-2/3 code $\mathbf{H}(D) = [1\ 1{+}D\ 1{+}D^2]$. A trellis for this type of encoder has a state complexity of 2^M states and a branch complexity of 2^{N-1} branches per state [2]. For such encoders, decoding algorithms such as trellis-based symbol-by-symbol maximum a-posteriori probability (MAP) decoding [7] cause a large computational burden for increasing rate due to the branch complexity. This questions the use of such encoders, since other encoders, e.g., in controller canonical form or punctured convolutional codes [2], both with much lower branch complexity, require less computational burden given the same state complexity. However, recent literature shows that some of the most favorable decoding algorithms such as MAP decoding can be performed using the trellis of the rate-$1/N$ dual code $\mathcal{C}^\perp$ to $\mathcal{C}$ [8], including approximate versions [9]. The branch complexity of the trellis of $\mathcal{C}^\perp$ is 2, which is the same as for high-rate codes punctured from rate-$1/N$ mother codes.

We seek to find codes $\mathbf{H}(D)$ with optimal spectral coefficents a_d, $d = 0, 1, \ldots$, defined in [2]. The sequence $\{a_d\}$ of the a_d is called spectrum and the smallest non-zero d for which $a_d \neq 0$ is called free distance d_f of the code. We call a code optimal with respect to a given memory M if its first d' spectral coefficents a_d, $d = 1, 2, \ldots, d'$ are lower than that of all other codes from the specified code class for a maximum d', e.g., the spectrum $\{1, 0, 0, 4, 8, \ldots\}$ is superior to $\{1, 0, 0, 5, 0, \ldots\}$.

In the following we analyze in Section II the properties of the considered class of codes, in particular the spectral coefficients a_2, a_3, and a_4 to derive efficient search strategies for optimal codes in Section III evolving in tables of found codes presented in the appendix. Table I lists examples of rate-$(N{-}1)/N$ encoders with improved spectrum compared with known codes at similar rates reported in [2], [3], [5], [10]

II. Analysis of Code Properties

We begin by presenting some definitions that will help in describing the code properties and search techniques. Code sequences $\mathbf{y}(D)$ of weight w_y are generated by information sequences $\mathbf{x}(D)$ of less or equal weight, since the encoder $\mathbf{G}(D)$ is systematic. A weight w_x information sequence $\mathbf{x}(D)$ can also be represented by

$$\mathbf{x}(D) = \sum_{j=1}^{w_x} \mathbf{u}_{t_j} D^{p_j}, \quad t_j \in \{1, \ldots, N{-}1\}, \quad p_j \geq 0,$$

TABLE I
Examples of rate-$(N-1)/N$ encoders with improved spectrum.

R	M	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}
8/9	3	17, 17 15 13 11 7 5 6 2	2	1,35
9/10	3	13, 17 15 13 11 7 5 6 2 15	2	2,43
10/11	3	17, 17 15 13 11 7 5 6 2 15 13	2	3,52
11/12	3	11, 17 15 13 11 7 5 6 2 17 15 13	2	4,62
8/9	4	17, 37 35 33 31 27 25 23 21	3	1,142
9/10	4	15, 37 35 33 31 27 25 23 21 13	3	4,234
10/11	4	17, 37 35 33 31 27 25 23 21 15 13	3	9,369
11/12	4	11, 37 35 33 31 27 25 23 21 17 15 13	3	16,547
16/17	5	37, 77 75 73 71 67 65 63 61 57 55 53 51 47 45 43 41	3	1,1100

where $\mathbf{u}_i$ is a $1\times K$ unit vector with a one at position i and any $t_j = t_{j'}$ implies that $p_j \neq p_{j'}$. We require at least one p_j to be zero yielding $\mathbf{x}_0 \neq 0$. Encoding $\mathbf{x}(D)$ to $\mathbf{y}(D)$ yields that

$$\mathrm{w}_H(\mathbf{y}(D)) = w_x + \mathrm{w}_H\left(\sum_{j=1}^{w_x} g_{t_j}(D)/g_0(D) \cdot D^{p_j}\right).$$

To have $\mathrm{w}_H(\mathbf{y}(D)) = w_y$, the term $\sum_{j=1}^{w_x} g_{t_j}(D)/g_0(D) \cdot D^{p_j}$ must be of the form $\sum_{j=1}^{w_y-w_x} D^{q_j}$ for some integer q_j being pairwise different. Multiplying with $g_0(D)$ and combining the sums yields

$$\sum_{j=1}^{w_x} g_{t_j}(D) D^{p_j} + \sum_{j=1}^{w_y-w_x} g_0(D) D^{q_j} = 0, \tag{1}$$

where the addition is modulo 2.

Definition 1: Let $\mathcal{W}_{w_y}$ be a set specifying the events that the encoder generates a weight w_y code sequence $\mathbf{y}(D)$. The set $\mathcal{W}_{w_y}$ contains all tuples

$$(t_1, ..., t_{w_x}; p_1, ..., p_{w_x}; q_1, ..., q_{w_y-w_x}), \quad 1 \leq w_x \leq w_y,$$

for which (1) subject to the constraints holds.

Lemma 1: The size $|\mathcal{W}_w|$ of $\mathcal{W}_w$ does not change if any generator $g_i(D)$ in $\mathbf{H}(D)$ is replaced by $g_i(D)D^{-r}$, $r > 0$, given that $g_{i,0} = ... = g_{i,r} = 0$.

Proof: If $g_0(D)$ in (1) is replaced with $g_0(D)D^{-r}$, the tuples $(t_j; p_j; q_1, ..., q_{w_y-w_x})$ are uniquely mapped to the new valid tuples $(t_j; p_j; q_1+r, ..., q_{w_y-w_x}+r)$. If $g_{t_j}(D)$ is replaced with $g_{t_j}(D)D^{-r}$ in (1), the tuples $(t_j; p_1, ..., p_j, ..., p_{w_x}; q_j)$ are uniquely mapped to the new valid tuples $(t_j; p_1, ..., p_j+r, ..., p_{w_x}; q_j)$ unless p_j was the only zero exponent. In latter case, the new tuples are invalid but there is the same amount of extra tuples $(t_j; p_1-r', ..., p_j+r-r', ..., p_{w_x}-r'; q_j)$, where r' is chosen such that some $p_{j'}$, $j = 1, ..., w_x$, is zero. Hence, the overall number of tuples, i.e., $|\mathcal{W}_w|$, is invariant. ■

Lemma 2: The size $|\mathcal{W}_w|$ of $\mathcal{W}_w$ is equal to the spectral coefficient a_w if the free distance d_f of the code is at least $\lfloor w/2 \rfloor + 1$.

Proof: The spectral coefficient a_w is the number of weight w code sequences encoded from some $\mathbf{x}(D)$ where $\mathbf{x}_0 \neq 0$, whose paths in the code trellis depart and approach the all-zero state only once. Latter constraint is not fulfilled by (1), which addresses all weight w code sequences where $\mathbf{x}_0 \neq 0$, but the code path can depart and rejoin the all-zero state many times. In such a case, the weight w code sequence addressed by (1) is the concatenation of two or more code sequences of less weight which depart and rejoin the all-zero state only once. Hence, this code sequence is not counted towards a_w. Also, the weight of this code sequence is the sum of the weights of the subsequences. However, if $d_f \geq (\lfloor w/2 \rfloor + 1)$, a weight w code sequence cannot be split into subsequences of less weight as described above, since $2(\lfloor w/2 \rfloor + 1) > w$. ■

With these two lemmas at hand, we are now ready to state the following theorem, which plays a crucial role in limiting the search requirements.

Theorem 1: There exists a rate-$(N{-}1)/N$ code $\mathcal{C}$ with $d_f \geq 3$ if and only if $R \leq (2^M{-}1)/2^M$. Above that rate, i.e. $N > 2^M$, the spectral coefficient a_2 is at least

$$a_2 \geq (2^M - (N \bmod 2^M))\binom{\lfloor N/2^M \rfloor}{2} + (N \bmod 2^M)\binom{\lfloor N/2^M \rfloor + 1}{2}.$$

Proof: Codes for which all $g_i(D)$ are non-zero achieve $d_f \geq 2$ and thus $|\mathcal{W}_2| = a_2$ by Lemma 2. Using Lemma 1 we can consider only those codes whose generators $g_i(D)$ have $g_{i,0} = 1$, $\forall i$, since codes with $g_i(D)$, $g_{i,0} = 0$, for some i have the same $|\mathcal{W}_2|$. For such codes, (1) holds if and only if two $g_i(D)$ are identical and $p_j = q_j = 0$, i.e., $\mathcal{W}_2$ contains only tuples of type $(t_1; 0; 0)$ or $(t_1, t_2; 0, 0; -)$.

($\Rightarrow$) A code achieving $d_f \geq 3$ has $|\mathcal{W}_2| = 0$, which is possible only with distinct generators $g_i(D)$ under the restriction that $g_{i,0} = 1$. There are 2^M distinct polynomials $g_i(D)$ up to degree M with $g_{i,0} = 1$. Since we need N $g_i(D)$ to construct a code $\mathbf{H}(D)$ of rate $(N-1)/N$, the largest possible rate to have distinct $g_i(D)$ with $g_{i,0} = 1$ in $\mathbf{H}(D)$ is $(2^M-1)/2^M$.

($\Leftarrow$) Given a rate $R \leq (2^M-1)/2^M$, there is a code $\mathbf{H}(D)$ with distinct $g_i(D)$, $g_{i,0} = 1$, and degree at most M yielding $|\mathcal{W}_2| = 0$. This implies that $d_f \geq 3$.

For the case where $R > (2^M{-}1)/2^M$, some $g_i(D)$, $g_{i,0} = 1$, occur more the once. We assume that each of the 2^M distinct $g_i(D)$ occur $n_k = 0, 1, ...,$ $k = 1, ..., 2^M$, times, such that $\mathbf{H}(D)$ contains $\sum_{k=1}^{2^M} n_k = N$, $N > 2^M$, generators yielding rate-$(N{-}1)/N$. Any pair of identical $g_i(D)$ in $\mathbf{H}(D)$ increases $|\mathcal{W}_2|$ by 1. Any triple of identical $g_i(D)$ increases $|\mathcal{W}_2|$ by 3, since there are 3 choices to select a pair from this triple increasing $|\mathcal{W}_2|$ by 1. In general, an n-tuple of identical generators increases $|\mathcal{W}_2|$ by $\binom{n}{2}$. It follows that the total size of $|\mathcal{W}_2|$ and thus a_2 is given by $\sum_{k=1}^{2^M}\binom{n_k}{2}$. Assume that any two $g_i(D)$ occurring n_k and $n_{k'}$ times in $\mathbf{H}(D)$ contribute $n_k + n_{k'} = \Delta N$ to the overall number N of generators. Their contribution to a_2 is $\binom{n_k}{2} + \binom{\Delta N - n_k}{2}$, which is minimized by $n_k = \lfloor \Delta N/2 \rfloor$. Thus, a_2 is minimized by pairwise "equalizing" the n_k, $\forall k$. This means we set all 2^M n_k to $\lfloor N/2^M \rfloor$ yielding $\sum_{k=1}^{2^M} n_k = 2^M \lfloor N/2^M \rfloor$. When $2^M \lfloor N/2^M \rfloor < N$, i.e., $(N \bmod 2^M) > 0$, we increase $(N \bmod 2^M)$ of the n_k by 1 to $\lfloor N/2^M \rfloor + 1$ yielding $\sum_{k=1}^{2^M} n_k = N$. Any code $\mathbf{H}(D)$ whose n_k are set according to this scheme achieve the minimal a_2 stated in the Theorem. ■

The construction outline in the proof above is used in Section III to restrict the number of possible codes achieving the desired optimal spectrum for rates above $(2^M-1)/2^M$.

Definition 2: Consider a set of n distinct polynomials $\{a_i(D)\}$, $i = 1, ..., n$, where $a_i(D) = 1 + a_{i,1}D + ... + a_{i,m_i}D^{m_i}$, $(a_{i,0} = 1)$, $a_{i,j} \in \mathbb{F}_2$, and $\deg(a_i(D)) = m_i$. Let $\mathcal{P}_m(a_1(D), ..., a_n(D))$ be the set of pairs $(a_i(D); a_{i'}(D)D^r)$, $i, i' \in \{1, ..., n\}$, $r > 0$, satisfying $\deg(a_i(D) + a_{i'}(D)D^r) \leq m$ and $m > m_i$, $\forall i$, which implies that $r = 1, 2, ..., m - m_{i'}$. Let $p_{i,i',r}(D)$ be the polynomial $a_i(D) + a_{i'}(D)D^r$ corresponding to the pair $(a_i(D); a_{i'}(D)D^r)$. The size of this set is

$$|\mathcal{P}_m(a_1(D), ..., a_n(D))| = n \cdot \sum\nolimits_{i=1}^{n}(m - m_i)$$

and it contains n^2 pairs $(a_i(D); a_{i'}(D)D^r)$ for which $p_{i,i',r}(D)$ has degree m, i.e., $r = m - m_{i'}$. For example, $\mathcal{P}_2(1, 1+D)$ is given by

$$\{(1; 1\cdot D), (1; 1\cdot D^2), (1; (1+D)\cdot D), (1+D; 1\cdot D), (1+D; 1\cdot D^2), (1+D; (1+D)\cdot D)\}.$$

Lemma 3: Among the n^2 $p_{i,i',r}(D)$ of degree m in $\mathcal{P}_m(a_1(D), ..., a_n(D))$, at least n are distinct.

Proof: Assume that the polynomials $b_k(D)$, $k\in\{1,...,x\}$, correspond to the x distinct $p_{i,i',r}(D)$ of degree m. Each $b_k(D)$ is occurring n_k times among all $p_{i,i',r}(D)$ of degree m such that $\sum_{k=1}^{x} n_k = n^2$.

If $x < n$, there must exist some n_k, which are larger than n, since otherwise $\sum_{k=1}^{x} n_k = n^2$ cannot hold. Given a particular $a_i(D)$ from the set $\mathcal{P}_m(a_1(D),...,a_n(D))$, the pair $(a_i(D); a_{i'}(D)D^r)$ yields $b_k(D)$ only if $a_{i'}(D) = b_k(D) - a_i(D))D^{m_{i'}-m}$, since r is restricted to $m - m_{i'}$. Thus, fixing $a_i(D)$ also fixes $a_{i'}(D)$ if such an $a_{i'}(D)$ exists at all. Since there are only n distinct $a_{i'}(D)$, at most n pairs $(a_i(D); a_{i'}(D)D^r)$ result in the same $b_k(D)$ and thus $n_k \leq n$ which implies that $x \geq n$. ■

Theorem 2: There exists a rate-$(N-1)/N$ code $\mathcal{C}$ with $d_f \geq 4$ if and only if $R \leq (2^{M-1} - 1)/2^{M-1}$.

Proof: We consider only those codes where all generators $g_i(D)$ have $g_{i,0} = 1$, $\forall i$, and which are distinct. This implies that $d_f \geq 3$ and $|\mathcal{W}_3| = a_3$ due to Theorem 1 and Lemma 2. For such codes, (1) holds only if exactly two of the p_j or q_j are zero and exactly one of the p_j or q_j is positive, since otherwise the scalar coefficients $g_{t_j,0}$ or $g_{0,0}$ do not cancel out. Thus, $\mathcal{W}_w$ contains only tuples of type $(t_1; 0; 0, q_2 > 0)$, $(t_1, t_2; 0, 0; q_1 > 0)$, or $(t_1, t_2, t_3; 0, 0, p_3 > 0; -)$. Equivalently, a tuple in $\mathcal{W}_w$ corresponds to a generator triple $(g_i(D), g_{i'}(D), g_{i'}(D)D^r)$, $r > 0$, $i, i' \in \{0,...,N-1\}$, for which $g_i(D) = g_{i'}(D) + g_{i'}(D)D^r$.

($\Leftarrow$) There is no generator triple yielding $g_i(D) = g_{i'}(D) + g_{i'}(D)D^r$ if all generators are distinct and have exactly degree M. A rate-$(N-1)/N$ code $\mathbf{H}(D)$ which contains N such $g_i(D)$, i.e. $g_{i,0} = g_{i,M} = 1$, achieves $|\mathcal{W}_3| = 0$ and thus $d_f \geq 4$. Since there are 2^{M-1} distinct polynomials $g_i(D)$ with $g_{i,0} = 1$ and degree M, for any rate up to $(2^{M-1} - 1)/2^{M-1}$ we can construct a code having $d_f \geq 4$.

($\Rightarrow$) Assume a code of rate above $(2^{M-1}-1)/2^{M-1}$ consisting of $N > 2^{M-1}$ generators. Let n of these have degree less than M. Since there are 2^{M-1} distinct $g_i(D)$ of degree M, n is lower bounded by $N - 2^{M-1}$. There might exist generator triples yielding $g_i(D) = g_{i'}(D) + g_{i'}(D)D^r$ where $g_{i'}(D)$ is one of those generators of degree less than M. In fact, if any $g_i(D)$ of degree M is equal to a $p_{i,i',r}(D)$ corresponding to a pair in the set $\mathcal{P}_M(\{g_{i'}\})$ constructed on the n generators of degree less than M, $\mathcal{W}_3$ is non-empty and thus $d_f = 3$. By Lemma 3, there are at least n distinct $p_{i,i',r}(D)$ of degree M. Thus, out of the N $g_i(D)$ in the code, n', where $n' \geq n$, of degree M must be excluded to assure that $\mathcal{W}_3$ stays empty. This decreases the rate to $(N-1-n')/(N-n')$. From the lower bound $n \leq (N - 2^{M-1})$ follows that the largest rate to achieve $d_f \geq 4$ is $(2^{M-1} - 1)/2^{M-1}$, which is achieved when $n = n'$. ■

It can be shown that for rates higher than $(2^{M-1} - 1)/2^{M-1}$ and up to $(2^M - 1)/2^M$, i.e. $2^M \geq N > 2^{M-1}$, the spectral coefficient a_3 is at least

$$a_3 \geq (N - 2^{M-1}) \cdot \sum_{i=0}^{N-1} (M - \deg(g_i(D)), \tag{2}$$

where $g_{i,0} = 1$, $\forall i$. It can also be shown that equality can be achieved constructively similar to Theorem 1. This significantly reduces the search space as illustrated in the next section. Note that traditional upper bounds on d_f, e.g., the Heller bound [2], are not easily applicable and do not provide constructive arguments to help limit the search.

III. Code search

The findings from Section II can be used to efficiently search codes $\mathbf{H}(D)$ with optimal spectrum given N and M. We do not distinguish equivalent codes, i.e., codes with identical spectrum. In particular, any permutation on the generators $g_i(D)$, $i = 0,...,N-1$, in $\mathbf{H}(D)$ yields an equivalent code, which can be encoded using an encoder of type $\mathbf{G}(D)$ when the recursive part, the rightmost entry of $\mathbf{H}(D)$, is non-zero. Latter constraint does not affect the search, since any zero generator would yield a poor code with $d_f = 1$.

For rates $(2^M-1)/2^M < R$ yielding $d_f = 2$ according to Theorem 1, the search strategy is as follows:

(A1) The minimal a_2 follows readily from Theorem 1 and only (N mod 2^M) of the N generators in $\mathbf{H}(D)$ can be chosen by selecting some of the 2^M distinct polynomials $g_i(D)$ having $g_{i,0} = 1$ up to degree M. From all possible codes we searched for those minimizing a_3.

(A2) From the codes found in (A1), those optimizing the 5 spectral coefficients a_d, $d = 4, ..., 8$, were obtained. The search included also codes derived from the ones found in (A1), where the generators of smaller degree than M where allowed to shift to degree M. For example, besides a code $\mathbf{H}(D) = [1\ 1\ 1+D]$ found in (A1), we also tested the spectrum of $\mathbf{H}'(D) = [1\ D\ 1+D]$ and $\mathbf{H}''(D) = [D\ D\ 1+D]$.

(A3) From the codes found in (A2), we searched for those codes optimizing $\{c_d\}$, $d = d_f, ..., 8$, where c_d is the total number of non-zero information bits generating weight d code sequences. The search included N encoders per found code, which was to select any of the N $g_i(D)$ as recursive part.

For rates $R \leq (2^M-1)/2^M$ yielding $d_f = 3$ according to Theorem 1, the search strategy is as follows:

(B1) The minimal $a_3 = 0$ for $R \leq (2^{M-1}-1)/2^{M-1}$ follows from Theorem 1. For $R > (2^{M-1}-1)/2^{M-1}$, a_3 follows from (2). We searched for codes achieving a_3, where all $g_i(D)$ have $g_{i,0} = 1$.

(B2) Similar to (A2).

(B3) Similar to (A3).

Figure 2 depicts the number of tested codes for various rates and memories. The upper curve in each plot corresponds to the case with no constraints on the search space. Using the results from Section II, the computational requirements are significantly lowered as shown by the lower curves in each plot. We note that there is room for improvement, which is currently being investigated.

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Appendix

The Tables II-VII show the found codes of rates 1/2 to 19/20 for memories $M = 1, 2, ..., 6$ together with the free distance d_f and the spectral coefficients a_{d_f}, a_{d_f+1} and c_{d_f}, c_{d_f+1}. Note that the search for memory 6 codes was at the time of the submission of this report still in progress.

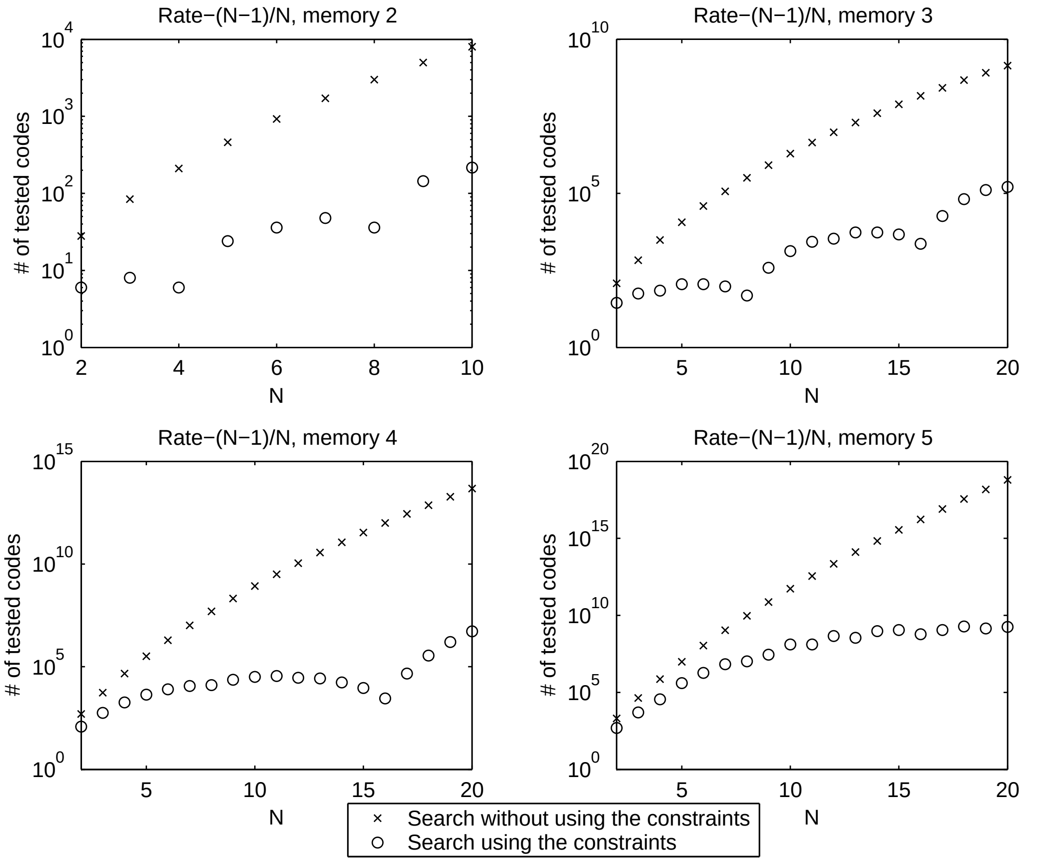

Fig. 2. Computational requirements in the search for rate-$(N-1)/N$, memory $2, 3, 4, 5$ codes.

R	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}	c_{d_f}, c_{d_f+1}
1/2	1, 3	3	1,1	1,2
2/3	3, 1 3	2	1,2	1,5
3/4	1, 1 3 3	2	2,8	3,16
4/5	3, 1 3 1 3	2	4,12	6,32
5/6	1, 1 3 1 3 3	2	6,27	10,63
6/7	3, 1 3 1 3 1 3	2	9,36	15,99
7/8	1, 1 3 1 3 1 3 3	2	12,64	21,160
8/9	3, 1 3 1 3 1 3 1 3	2	16,80	28,224
9/10	1, 1 3 1 3 1 3 1 3 3	2	20,125	36,325

TABLE II

Rate-$(N-1)/N$, memory 1 codes with optimal spectrum.

R	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}	c_{d_f}, c_{d_f+1}
1/2	5, 7	5	1,2	2,6
2/3	3, 7 5	3	1,4	1,10
3/4	3, 7 5 2	3	6,23	13,68
4/5	7, 7 5 3 2	2	1,9	1,24
5/6	5, 7 5 3 2 7	2	2,13	3,35
6/7	3, 7 5 3 2 7 5	2	3,24	5,59
7/8	3, 7 5 3 2 7 5 2	2	4,48	7,124
8/9	7, 7 5 3 2 7 5 3 2	2	6,60	10,168
9/10	5, 7 5 3 2 7 5 3 2 7	2	8,74	14,208

TABLE III

RATE-$(N-1)/N$, MEMORY 2 CODES WITH OPTIMAL SPECTRUM.

R	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}	c_{d_f}, c_{d_f+1}
1/2	17, 15	7	1,3	2,12
2/3	17, 15 13	5	1,5	2,20
3/4	11, 17 15 13	4	5,36	15,128
4/5	7, 17 15 13 11	3	1,21	1,64
5/6	5, 17 15 13 11 7	3	4,53	8,174
6/7	5, 17 15 13 11 7 6	3	12,124	29,424
7/8	5, 17 15 13 11 7 6 2	3	28,274	73,956
8/9	17, 17 15 13 11 7 5 6 2	2	1,35	1,98
9/10	13, 17 15 13 11 7 5 6 2 15	2	2,43	3,121
10/11	17, 17 15 13 11 7 5 6 2 15 13	2	3,52	5,147
11/12	11, 17 15 13 11 7 5 6 2 17 15 13	2	4,62	7,176
12/13	7, 17 15 13 11 7 5 6 2 17 15 13 11	2	5,83	9,226
13/14	5, 17 15 13 11 7 5 6 2 17 15 13 11 7	2	6,109	11,299
14/15	5, 17 15 13 11 7 5 6 2 17 15 13 11 7 6	2	7,154	13,427
15/16	5, 17 15 13 11 7 5 6 2 17 15 13 11 7 6 2	2	8,224	15,628
16/17	17, 17 15 13 11 7 5 6 2 17 15 13 11 7 5 6 2	2	10,252	18,728
17/18	13, 17 15 13 11 7 5 6 2 17 15 13 11 7 5 6 2 15	2	12,282	22,816
18/19	17, 17 15 13 11 7 5 6 2 17 15 13 11 7 5 6 2 15 13	2	14,314	26,910
19/20	11, 17 15 13 11 7 5 6 2 17 15 13 11 7 5 6 2 17 15 13	2	16,348	30,1010

TABLE IV

RATE-$(N-1)/N$, MEMORY 3 CODES WITH OPTIMAL SPECTRUM.

R	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}	c_{d_f}, c_{d_f+1}
1/2	35, 23	7	2,3	6,12
2/3	35, 25 23	5	2,13	6,52
3/4	35, 31 27 23	4	1,16	3,60
4/5	33, 31 27 23 35	4	7,56	20,224
5/6	25, 37 35 31 27 23	4	19,160	62,660
6/7	33, 31 27 25 23 37 35	4	43,351	144,1512
7/8	21, 37 35 33 31 27 25 23	4	78,784	273,3368
8/9	17, 37 35 33 31 27 25 23 21	3	1,142	1,496
9/10	15, 37 35 33 31 27 25 23 21 13	3	4,234	8,834
10/11	17, 37 35 33 31 27 25 23 21 15 13	3	9,369	21,1328
11/12	11, 37 35 33 31 27 25 23 21 17 15 13	3	16,547	40,1995
12/13	11, 37 35 33 31 27 25 23 21 17 16 15 13	3	30,824	79,3037
13/14	17, 37 35 33 31 27 25 23 21 16 15 13 12 11	3	48,1195	131,4413
14/15	11, 37 35 33 31 27 25 23 21 17 16 15 13 12 6	3	77,1764	213,6585
15/16	11, 37 35 33 31 27 25 23 21 17 16 15 13 12 6 4	3	120,2644	337,9908
16/17	37, 37 35 33 31 27 25 23 21 17 15 13 11 16 12 6 4	2	1,135	1,390
17/18	35, 37 35 33 31 27 25 23 21 17 16 15 13 12 11 6 4 27	2	2,151	3,437
18/19	33, 37 35 33 31 27 25 23 21 17 16 15 13 11 12 6 4 37 25	2	3,168	5,487
19/20	31, 37 35 33 31 27 25 23 21 17 16 15 13 12 11 6 4 35 27 23	2	4,186	7,540

TABLE V

Rate-$(N-1)/N$, memory 4 codes with optimal spectrum.

R	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}	c_{d_f}, c_{d_f+1}
1/2	53, 75	8	1,8	4,34
2/3	55, 65 43	6	6,27	24,125
3/4	43, 77 71 53	5	7,45	25,204
4/5	51, 75 47 41 33	4	1,36	3,142
5/6	41, 75 57 51 47 33	4	5,96	16,399
6/7	21, 75 73 67 55 53 47	4	13,223	44,940
7/8	43, 73 71 67 65 57 55 42	4	28,447	96,1962
8/9	65, 77 75 73 63 51 45 41 56	4	55,812	192,3612
9/10	57, 71 67 65 61 55 53 45 43 37	4	95,1394	338,6306
10/11	45, 77 75 71 67 61 55 53 43 46 51	4	151,2260	543,10305
11/12	53, 77 75 73 65 51 47 45 41 27 63 61	4	223,3553	814,16308
12/13	65, 76 63 61 55 53 51 47 45 43 75 71 67	4	328,5316	1201,24587
13/14	65, 63 61 57 55 53 51 47 45 43 37 75 71 67	4	456,7691	1687,35769
14/15	73, 71 67 65 63 61 57 55 53 51 47 45 43 37 75	4	621,10873	2308,50837
15/16	41, 77 75 73 71 67 65 63 61 57 55 53 51 47 45 43	4	844,14400	3165,67056
16/17	37, 77 75 73 71 67 65 63 61 57 55 53 51 47 45 43 41	3	1,1100	1,4128
17/18	27, 77 75 73 71 67 65 63 61 57 55 53 51 47 45 43 41 35	3	4,1408	8,5306
18/19	33, 77 75 73 71 67 65 63 61 25 57 55 53 51 47 45 43 41 37	3	9,1790	21,6758
19/20	23, 77 75 73 71 67 65 63 61 57 55 53 51 47 45 43 41 35 31 27	3	16,2237	40,8479

TABLE VI

RATE-$(N-1)/N$, MEMORY 5 CODES WITH OPTIMAL SPECTRUM.

R	$g_0(D), g_1(D)...g_{N-1}(D)$	d_f	a_{d_f}, a_{d_f+1}	c_{d_f}, c_{d_f+1}
1/2	171, 133	10	11,0	50,0
2/3	105, 163 145	7	17,53	78,276
3/4	155, 161 135 103	6	27,118	118,614
4/5	123, 177 155 145 107	5	11,100	43,476
5/6	135, 176 163 147 131 105	5	48,360	200,1798
6/7	145, 163 141 135 131 113 77	4	3,99	10,423
7/8	111, 176 173 151 145 135 121 103	4	9,209	29,911
8/9	141, 173 167 153 147 135 125 111 43	4	19,407	65,1811
9/10		4		
10/11		4		
11/12		4		
12/13		4		
13/14		4		
14/15		4		
15/16		4		
16/17		4		
17/18		4		
18/19		4		
19/20		4		

TABLE VII

Rate-$(N-1)/N$, memory 6 codes with optimal spectrum.