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The Multiple Knapsack Problem With Color Constraints

Milind Dawande Jayant Kalagnanam

IBM Research Division
T.J. Watson Research Center
Yorktown Heights, New York 10598

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Milind Dawande Jayant Kalagnanam

IBM, T. J. Watson Research Center
Yorktown Heights, NY 10598

Abstract

In this paper, we study a new problem which we call the *multiple knapsack problem with color constraints (MKCP)*. Motivated by a real application from the steel industry [6], we generalize the multiple knapsack problem by adding a new attribute (called "color" in the paper) to an item and then adding the associated "color" constraints which restrict the number of distinct colors which can be assigned to a knapsack. The color constraints provide a representation for a general class of processing constraints.
We consider the MKCP from a polyhedral point of view. Three classes of facet defining inequalities and four classes of other valid inequalities are discussed. We establish necessary and sufficient conditions under which a color constraint is redundant and also provide separate necessary conditions and sufficient conditions under which a color constraint is facet defining. In addition, we prove that a basic fractional solution to the linear programming relaxation can be eliminated by a facet defining inequality associated with an individual knapsack constraint.

Key Words: multiple knapsack problem; color constraints; polyhedral structure, valid inequalities, facets.

1 Introduction

In this paper we study a multiple knapsack problem with color constraints (MKCP), a formulation derived for a real application in the steel industry [6]. The conventional multiple knapsack (MKP) problem considers a set of N items which need to be packed into M knapsacks to maximize profit. We generalize this conventional problem by associating with each item a color attribute and restricting the number of colors in any knapsack to be less than two. Although the number of allowable colors in any knapsack can in general be any $K \leq N$, the results in this paper are based on allowing no more than two colors in any knapsack. However, some of the results extend to the case when more than two colors are allowed for a knapsack. Additional assignment restrictions are specified in terms of a subset of knapsacks which can hold any given item.

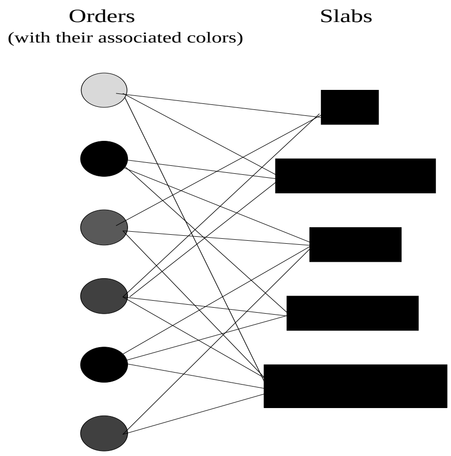

Figure 1: The multiple knapsack problem with colors

The multiple knapsack problem with color constraints is motivated by our experience with the *surplus inventory matching problem* in the steel industry [6]. Production planning begins with an order book which contains a list of orders that need to be satisfied. Before planning production, an attempt is made to satisfy orders using leftover slabs from surplus inventory. The problem of applying orders against an existing surplus inventory is called the surplus inventory matching problem and can be formulated as a multiple knapsack problem with color and assignment constraints. The goal of inventory matching is to maximize the total weight of the order book that is applied against existing inventory. In the following paragraphs we explain the background for the assignment and color constraints. For pedagogical purposes we describe the assignment restrictions before the color constraints.

For each order in the order book we can identify a set of applicable slabs from the surplus inventory. These assignment restrictions are based on quality and physical dimension considerations. For any given order only slabs which are of the same quality or better can be applied. In addition, the thickness and width requirements for each order need to be compatible with those of each applicable slab. These considerations restrict the number of applicable slabs for each order. The surplus inventory matching problem can be represented by a undirected bipartite graph as shown in Figure 1.

The assignment constraints can also be thought of as specifying a list of orders that can be applied against any slab in the surplus inventory. However, not all orders assignable to a slab can be packed together on the slab. Such packing constraints emerge out of processing considerations in the finishing line of a steel mill. Associated with each order is a route that specifies the set of process operations that need to be applied in the finishing mill. Orders with different routes require different process operations and are referred to as being of different *type*. Slabs packed with different orders types need to be cut before they are processed in the finishing mill. Since cutting slabs is expensive and often the cutting machine is a bottleneck, strong constraints are posed in terms of the number of allowed cuts per slab. The simplest and most commonly used constraint is to limit the number of required cuts to one, i.e. no more that two order types are allowed on a slab. In order to describe this constraint formally we associate a unique *color* for each route code and restrict the number of colors on a slab to be no more than two. Notice that this implies that we associate a color with each order based on its route code. This restricts the number of different order types on a slab to two and the number of required cuts to be no more than one.

The rest of the paper is organized as follows. The following section provides two formulations for the MKCP. Section 3 shows that the dimension of MKCP is the same as the multiple knapsack problem and examines which facets of MKP are also facets for MKCP. In addition, it is shown that the color constraints are also facets of MKCP. Section 4 provides a strengthening of four classes of valid inequalities for MKP based on the special structure of MKCP. Section 5 examines the structure of basic fractional solutions to the linear programming relaxation of MKCP. It is shown that such a solution can be eliminated by a facet defining inequality associated with an individual knapsack constraint. Section 6 discusses, in brief, the extension when the number of distinct colors allowed for a knapsack is more than 2. Section 7 summarizes the results of this paper.

2 Formulations for MKCP

It is easy to see $MKCP$ is NP-hard since if all items have the same color and we have only one knapsack, the $MKCP$ reduces to the well known *subset sum problem*, which is known to be NP-hard [7]. In this section, we provide two formulations for MKCP. The first formulation is more natural, in the sense that it has a separate set of variables for modeling the color constraints of the problem. The second formulation implicitly models the color constraints without introducing new variables. The first formulation has more variables than the second formulation but has fewer constraints.

n	$=$	Total number of items.
m	$=$	Total number of knapsacks.
N	$=$	Set of items.
M	$=$	Set of Knapsacks.
N_i	$=$	Set of knapsacks incident to item i.
N_j	$=$	Set of items incident to knapsack j.
x_{ij}	$=$	1 if item i is assigned to some knapsack; 0 otherwise.
C_j	$=$	Set of colors incident on knapsack j.
y_j^c	$=$	1 if an item(s) of color c obtains material from knapsack j; 0 otherwise.
O_i	$=$	Weight of item i.
W_j	$=$	Capacity of knapsack j.

Table 1: List of notations

$$\text{Max} \quad \sum_{i=1}^{n} \sum_{j \in N_i} O_i x_{ij}$$

subject to

$$\begin{array}{lll}
\sum_{i \in N_j} O_i x_{ij} & \leq W_j & 1 \leq j \leq m \\
\sum_{j \in N_i} x_{ij} & \leq 1 & 1 \leq i \leq n \\
\sum_{c \in C_j} y_j^c & \leq 2 & 1 \leq j \leq m \\
x_{ij} & \leq y_j^{c(i)} & 1 \leq i \leq n, \ j \in N_i
\end{array}$$

$$
\begin{array}{lll}
x_{ij} & \in \{0,1\} & 1 \le i \le n,\ j \in N_i \\
y_j^c & \in \{0,1\} & \forall c \in C_j,\ 1 \le j \le m
\end{array}
$$

The first and second set of constraints are the well known knapsack constraints (for the knapsacks) and assignment constraints (for the items). The third and fourth set of constraints restrict the total number of colors incident on a knapsack to less than two. The total number of variables in this formulation is $\sum_{i=1}^{n} |N_i| + \sum_{j=1}^{m} |C_j|$ $(= \sum_{j=1}^{m} |N_j| + \sum_{j=1}^{m} |C_j|)$ while the total number of constraints is $\sum_{i=1}^{n} |N_i| + 2m + n$. In the formulation above, a separate set of variables (namely, the y_j^c variables) are used to model the color constraints. However, using a separate set of variables is not necessary. Below, we model the problem without introducing new variables.

$$
(P^*) \qquad \text{Max} \qquad \sum_{i=i}^{n} \sum_{j \in N_i} O_i x_{ij}
$$

subject to

$$
\begin{array}{lll}
\sum_{i \in N_j} O_i x_{ij} & \le W_j & 1 \le j \le m \\
\sum_{j \in N_i} x_{ij} & \le 1 & 1 \le i \le n \\
x_{i_1 j} + x_{i_2 j} + \ldots + x_{i_k j} & \le 2 & \forall \text{ maximal sets of size } k \ge 3 \\
 & & \text{of items of different colors } \forall\ j \\
x_{ij} & \in \{0,1\} & 1 \le i \le n,\ j \in N_i
\end{array}
$$

Note that $x_{ij} = 1$ implies that item i has been assigned to knapsack j. If the total number of distinct colors incident on a knapsack is less than or equal to 2, then, clearly, the color constraints are not required for the knapsack. If there are more than two colors incident on the knapsack j, then, of the $|N_j|$ items incident on knapsack j, let $|N_j^k|$ be the set of items of color k, $k = 1, \ldots, |C_j|$. For every maximal set of items (incident on knapsack j) having different colors, we add a constraint which states that the a maximum of two items can be assigned from the maximal set. A maximal set of items having different colors involves choosing exactly one item of each color. Hence, there are a total of $\Pi_{k=1}^{|C_j|}|N_j^k|$ constraints (one for each maximal set) for knapsack j. These constraints, together, state that the total number of different colors incident on knapsack j is bounded above by two. Such a constraint set is added for each knapsack having more than two colors incident on it. For this formulation, the total number of variables is $\sum_{i=1}^{n}|N_i|$ $(=\sum_{j=1}^{m}|N_j|)$. The total number of constraints is bounded above by $\sum_{j=1}^{m}\Pi_{k=1}^{|C_j|}|N_j^k|+m+n$.

We illustrate the above formulations on a small example.

Example 1: Consider Figure 2. We have four items I_1, I_2, I_3 and I_4 having weights 5, 8, 6 and 4 respectively, and two knapsacks, K_1 and K_2 having capacities 8 and 15 respectively. Items I_1 and I_2 have the same color (c_1) while items I_3 and I_4 have colors c_2 and c_3 respectively. Thus, the total number of distinct colors for the items is 3.

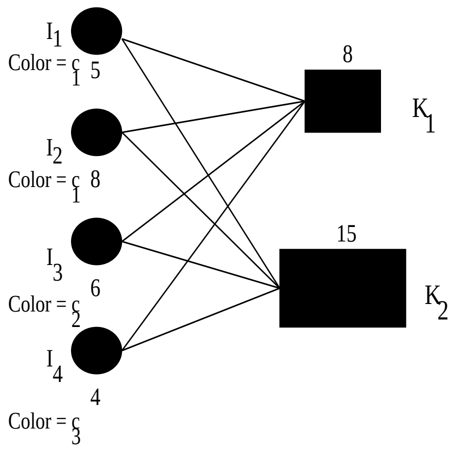

Figure 2: Example 1

__**Formulation 1**__:

$$\text{Max} \quad 5x_{11}+5x_{12}+8x_{21}+8x_{22}+6x_{31}+6x_{32}+4x_{41}+4x_{42}$$

subject to

$$5x_{11} + 8x_{21} + 6x_{31} + 4x_{41} \leq 8$$

$$5x_{12} + 8x_{22} + 6x_{32} + 4x_{42} \leq 15$$

$$x_{11} + x_{12} \leq 1, \quad x_{21} + x_{22} \leq 1$$

$$x_{31} + x_{32} \leq 1, \quad x_{41} + x_{42} \leq 1$$

$$y_1^{c_1} + y_1^{c_2} + y_1^{c_3} \leq 2, \quad y_2^{c_1} + y_2^{c_2} + y_2^{c_3} \leq 2$$

$$x_{11} \leq y_1^{c_1}, \quad x_{12} \leq y_2^{c_1}, \quad x_{21} \leq y_1^{c_1}, \quad x_{22} \leq y_2^{c_1}$$

$$x_{31} \leq y_1^{c_2}, \quad x_{32} \leq y_2^{c_2}, \quad x_{41} \leq y_1^{c_3}, \quad x_{42} \leq y_2^{c_3}$$

$$x_{ij} \in \{0, 1\}, \; y_j^c \in \{0, 1\}$$

Formulation 2:

$$\text{Max} \quad 5x_{11} + 5x_{12} + 8x_{21} + 8x_{22} + 6x_{31} + 6x_{32} + 4x_{41} + 4x_{42}$$

subject to

$$5x_{11} + 8x_{21} + 6x_{31} + 4x_{41} \leq 8$$

$$5x_{12} + 8x_{22} + 6x_{32} + 4x_{42} \leq 15$$

$$x_{11} + x_{12} \leq 1, \quad x_{21} + x_{22} \leq 1$$

$$x_{31} + x_{32} \leq 1, \quad x_{41} + x_{42} \leq 1$$

$$x_{11} + x_{31} + x_{41} \leq 2, \quad x_{21} + x_{31} + x_{41} \leq 2$$

$$x_{12} + x_{32} + x_{42} \leq 2, \quad x_{22} + x_{32} + x_{42} \leq 2$$

$$x_{ij} \in \{0, 1\}$$

3 Dimension and Facets

In the traditional multiple knapsack problem [4][2], an item can be assigned (if possible) to any knapsack. In other words, the bipartite graph, $G(V, E)$ (Figure 1) is complete. In the MKCP, typically, an item could be assigned to one knapsack among a subset of knapsacks. That is, $|N_i| \neq M$, in general. However, without loss of generality, we can assume that G is complete. For if not, a missing edge (i, j) can be added to the bipartite graph and the corresponding variable x_{ij} can be introduced into the formulations with objective function coefficient zero. With the above modification, we can assume $N_i = M \ \forall i \in N$ and $N_j = N \ \forall j \in M$. Call the modified problem (P). Then, it is easy to see that an optimal solution to (P) gives an optimal solution to (P^*) with the same objective function value. For simplicity, we assume that for every item i, $O_i \leq W_j \ \forall j \in M$. We refer to the convex hull of feasible 0-1 solutions to (P) as P_{MKCP}. The corresponding multiple knapsack polytope, obtained by removing the color restrictions, will be referred to as P_{MKP}. A single knapsack problem obtained by considering a knapsack $k \in M$ and a set of items $\bar{N} \subseteq N$ will be denoted by $SK(\bar{N}, \bar{O}, W_k)$ where $\bar{O}$ is the vector of item weights corresponding to items in $\bar{N}$.

Theorem 3.1 *Dimension of the P_{MKCP} is mn.*

Proof: Denote the dimension of P_{MKCP} by d. It is well known that the dimension of P_{MKP} is mn [4]. Note that every feasible solution to MKCP is also feasible for the corresponding MKP problem. Hence $d \leq mn$.

Now, the zero vector and the mn unit vectors are affinely independent and valid 0-1 solutions for P_{MKCP}. Hence, $d \geq mn$. The result follows. □

The following result is easy to show.

Theorem 3.2 *The upper bound constraint $x_{ij} \leq 1$ defines a facet for P_{MKCP} if and only if $O_{i^*} + O_i \leq W_j$ where $O_{i^*} = \max_{k \in N_j - \{i\}} O_k$.*

The polyhedral structure of single knapsack and multiple knapsack problems has been widely studied in the operations research literature [1][8][9][10][4][2]. A well known result, proved in [4], is that all non-trivial facets for the single knapsack problem are facets for the multiple knapsack problem. Given the proximity of the MKCP to the MKP, it is then natural to ask a similar question about which facets for the MKP are also facets for MKCP.

Definition: A facet $\pi x \leq \pi_0$ of P_{MKP} is said to be *color-feasible* for the corresponding P_{MKCP} if there exist mn affinely independent feasible 0-1 points $x^i,\ i = 1, ..., mn$ for P_{MKP} with $\pi x^i = \pi_0$ such that all these points also satisfy the color constraints.

Lemma 3.3 *A facet $\pi x \leq \pi_0$ of P_{MKP} is also a facet for P_{MKCP} if and only if it is color feasible for the corresponding P_{MKCP}.*

Proof: Suppose a facet $\pi x \leq \pi_0$ of P_{MKP} is also a facet for P_{MKCP}. Then there exists mn 0-1 points $x^i,\ i = 1, ..., mn$ which are feasible for P_{MKCP} such that $\pi x^i = \pi_0$. Note that each of these points is also feasible for P_{MKP}.

Conversely, suppose a facet $\pi x \leq \pi_0$ of P_{MKP} is color feasible for the corresponding P_{MKCP}. Since, $P_{MKCP} \subseteq P_{MKP}$, $\pi x \leq \pi_0$ is valid for P_{MKCP}. Also, there exists mn feasible 0-1 points $x^i,\ i = 1, ..., mn$ with $\pi x^i = \pi_0$ which are valid for P_{MKP} and also satisfy the color constraints. Thus, $x^i,\ i = 1, ..., mn$ are valid for P_{MKCP}. It then follows that $\pi x^i = \pi_0$ defines a facet for P_{MKCP}. $\square$

In [4] it has been shown that the non-negativity constraints $x_{ij} \geq 0,\ i \in M, j \in N$ and the assignment constraints $\sum_{j \in N(i)} x_{ij} \leq 1,\ i \in M$ are facets of P_{MKP}. Also, it is easy to see that these facets are color-feasible for P_{MKCP}. Then, Lemma 3.3 implies the following result.

Theorem 3.4 *The following inequalities are facets of P_{MKCP}.*

- *The nonnegativity constraints* $x_{ij} \geq 0,\ i \in M, j \in N$.

- *The assignment constraints* $\sum_{j \in N(i)} x_{ij} \leq 1,\ i \in M$.

3.1 The Color Constraints

Since the color constraints are fundamental to the MKCP, we first examine these constraints for the single knapsack problem with colors (SKCP) and then extend our results to the MKCP. For a single knapsack problem, let k color classes $1, 2, ..., k$ be incident on the knapsack. The color constraint, then, takes the form

$$x_{i_1} + x_{i_2} + ... + x_{i_k} \leq 2 \tag{1}$$

where item i_j has color j. For simplicity, we assume that the dimension of the convex hull, P_{SKCP}, of feasible integer solutions to SKCP is n. Let $S = \{i_1, i_2, ..., i_k\}$ represent the set of items in the color constraint (1). Define $S^c = N \backslash S$. Each pair of items (i_p, i_q) in S with $O_{i_p} + O_{i_p} \leq W$, where W is the weight of the knapsack, corresponds to a characteristic vector in $\{0,1\}^n$ which is feasible and satisfies (1) with equality. If no such pair exists, then, clearly, (1) is redundant. Below, we characterize this.

Theorem 3.5 *The color constraint (1) is redundant if and only if every collection of three items from S which does not appear in any other color constraint is a cover for the knapsack.*

Proof: Suppose every collection of three items from S which does not appear in any other color constraint is a cover. Consider any collection $C = \{i_1, i_2, i_3\}$ of three items from S. If C belongs to some other color constraint then, clearly, $x_{i_1} + x_{i_2} + x_{i_3} \leq 2$. If C does not appear in any other color constraint, then, by the hypothesis, C is a cover and hence $x_{i_1} + x_{i_2} + x_{i_3} \leq 2$. Thus, in any case, no more than two items from C can be assigned to the knapsack. This is the case for every collection of three items from S. Hence, we have $x_{i_1} + x_{i_2} + ... + x_{i_k} \leq 2$. Conversely, if the constraint (1) is redundant, then, for any collection $C = \{i_1, i_2, i_3\}$ of three items from S, we have $x_{i_1} + x_{i_2} + x_{i_3} \leq 2$. Hence, either $\{i_1, i_2, i_3\}$ is a cover or C appears as part of some other color constraint. □

On the other hand, if there exists n characteristic vectors tight for (1) such that the resulting $n \times n$ matrix of these vectors is nonsingular, the inequality (1) defines a facet for SKCP. Below, we present a simple sufficient condition under which the constraint (1) is a facet for SKCP. The condition stated in the theorem is weaker than what is actually required but we present it here since it is more intuitive and relates well to the standard knapsack terminology.

Theorem 3.6 *If, for every pair of items $\{i_p, i_q\}$ from S, the collection of three items $\{i_p, i_q, i_l\}$ is not a cover for every item $i_l \in S^c$, the constraint (1) defines a facet for SKCP.*

Proof: It is easy to see that the condition in the theorem enables us to demonstrate n feasible 0-1 points which satisfy (1) with equality. First, consider the following $k \times k$ matrix A whose columns are indexed by the items of S (Figure 3). Each row of A contains exactly two 1s. It is easy to see that A is non-singular. Now, we construct an $n \times n$ matrix A^* as follows (see Figure 4): Matrix A is a submatrix of A^* and occupies the first k rows and k columns. The remaining $n - k$ columns of A^* are indexed by items of S^c. For every item $i_p \in S^c$, since S is a *maximal* collection of items of different colors, there exists an item, say $i_t \in S$, such that items i_p and i_t have the same colors. Then, the vector $v_{i_l} \in \{0,1\}^n$ which has three 1s corresponding to items i_l, i_t and any other item $i_s \in S \backslash \{i_t\}$ is feasible due to the hypothesis. We construct such a vector for each item in S^c. These vectors correspond to the remaining $n - k$ rows of A^*. Since each of these $n - k$ rows added is linearly independent of the first k rows, it is easy to verify that the matrix A^* is non-singular. Also, note that each row of A^* is a 0-1 feasible point for SKCP which satisfies (1) with equality. The result follows. □

$$A_{k\times k} = \begin{array}{c} \begin{matrix} i_1 & i_2 & i_3 & i_4 & \cdots & & & i_k \end{matrix} \\ \begin{bmatrix} 1 & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & & 1 & 1 & & 0 & & \\ & & & & \ddots & & & \\ & & & & 1 & 1 & & \\ & 0 & & & & 1 & 1 & \\ & & & & & & 1 & 1 \\ & & & & & 1 & & 1 \end{bmatrix} \end{array}$$

Figure 3: The $k \times k$ matrix A in Theorem 3.4

$$A^*_{n\times n} = \begin{array}{c} \begin{matrix} & i_s & i_t & & & i_p & & & \end{matrix} \\ \left[\begin{array}{ccccc:cccc} \multicolumn{5}{c:}{A_{k\times k}} & \multicolumn{4}{c}{0} \\ \hdashline & 1 & 1 & & & 1 & & & \\ 1 & & & & 1 & & 1 & & \\ & & & & & & & 1 & \\ & 1 & & 1 & & & & & 1 \end{array}\right] \end{array}$$

Figure 4: The $n \times n$ matrix A^* in Theorem 3.4

As stated above, the sufficient condition provided in Theorem 3.6 is weaker. Below, we state, without proof, a stronger condition.

Theorem 3.7 *If*

1. *For every item $i_l \in S^c$, there exists a pair of items $\{i_p^l, i_q^l\}$ in S such that the collection $\{i_p^l, i_q^l, i_l\}$ is not a cover and*

2. *There exists k pairs of items $\{i_{p_l}, j_{q_l}\}$, $l = 1, ..., k$ from S such that the corresponding k dimensional characteristic vectors are linearly independent in $\Re^k$*

then (1) defines a facet for SKCP.

The following result, which is easy to prove, provides a simple necessary condition for the inequality (1) to define a facet of SKCP.

Theorem 3.8 *If (1) defines a facet for SKCP then for every item $i_l \in S^c$, there exists a pair of items $\{i_p^l, i_q^l\}$ in S such that the collection $\{i_p^l, i_q^l, i_l\}$ is not a cover.*

Now, it is easy to extend the above results to MKCP. For the MKCP, for a knapsack, say knapsack j, the color constraint takes the form

$$x_{i_1 j} + x_{i_2 j} + ... + x_{i_k j} \leq 2 \tag{2}$$

Define $S_j = \{i_1, i_2, ..., i_k\}$ and $S_j^c = N_j \backslash S_j$. Theorems 3.4 and 3.5 generalize to the MKCP under the same conditions.

Theorem 3.9 *The color constraint (2) is redundant for MKCP if and only if every collection of three items from S_j which does not appear in any other color constraint for knapsack j is a cover for the knapsack j.*

Theorem 3.10 *If, for every pair of items $\{i_p, i_q\}$ from S_j, the collection of three items $\{i_p, i_q, i_l\}$ is not a cover for every item $i_l \in S_j^c$, the constraint (2) defines a facet for MKCP.*

Theorem 3.11 *If (2) defines a facet for MKCP then for every item $i_l \in S_j^c$, there exists a pair of items $\{i_p^l, i_q^l\}$ in S such that the collection $\{i_p^l, i_q^l, i_l\}$ is not a cover for the knapsack j.*

4 Strengthening of Valid Inequalities

Since $P_{MKCP} \subseteq P_{MKP}$, all valid inequalities for P_{MKP} are valid for P_{MKCP}. So, the more interesting question is: Can these inequalities be strengthened for the special structure of

P_{MKCP}? Below, we consider some known classes of valid inequalities for P_{MKP}.

4.1 Minimal Cover Inequalities

At set $C \subseteq N$ is a *cover* with respect to some knapsack $k \in M$ if $\sum_{i \in C} O_i > W_k$. The cover is *minimal* with respect to k if $\sum_{i \in C \setminus \{c\}} O_i \leq W_k$ for all $c \in C$. Suppose that $C \subseteq N$ is a minimal cover with respect to some knapsack k. The inequality

$$\sum_{i \in C} x_{ik} \leq |C| - 1 \tag{3}$$

is called the *minimal cover inequality* corresponding to C and k. It was shown in [1],[5] and [9] that the minimal cover inequality corresponding to C and k defines a facet for $SK(C, O, W_k)$. [4] shows that all non-trivial facets for $SK(C, O, W_k)$ are facets for P_{MKP}.

Suppose the minimal cover C is composed of $p \leq N$ distinct colors, $c_1, c_2, ..., c_p$. Let $S(c_1), S(c_2), ..., S(c_p)$ be the sets of items, respectively, corresponding to the colors $c_1, c_2, ..., c_p$. Thus,

$$|C| = |S(c_1)| + |S(c_2)| + ... + |S(c_p)| \tag{4}$$

Without loss of generality, assume $|S(c_1)| \geq |S(c_2)| \geq ... \geq |S(c_p)|$. Then, no more that $|S(c_1)| + |S(c_2)|$ edges can be incident on knapsack k. For if not, then knapsack k will have more than two colors incident on it. Hence, the constraint

$$\sum_{i \in C} x_{ik} \leq \min\{|C| - 1, |S(c_1)| + |S(c_2)|\} \tag{5}$$

is valid for MKCP. Note that due to (4), the inequality (5) can be a considerable strengthening over the inequality (3). Specifically, whenever $p \geq 4$, the inequality (5) is stronger than (3).

It is instructive to study conditions under which the strengthened inequality (5) is facet defining for MKCP. The following result is easy to prove:

Lemma 4.1 *If $|S(c_1)| + |S(c_2)| \leq |C| - 1$ and if $\forall\ i \in N \backslash C$*

1. *item i has the same color as c_1 or c_2 and*

2. *$S(c_1) \cup S(c_2) \cup \{i\}$ is not a cover for knapsack k*

then the strengthened inequality (5) defines a facet for MKCP.

4.2 Multiple Cover Inequalities

A set of items $C \subseteq N$ with the property that $\sum_{i \in C} O_i > \sum_{j \in J} W_j$ for $J \subseteq M$ is called a *multiple cover* with respect to J. In [10], it was shown that the *multiple cover inequality*

$$\sum_{j \in J} \sum_{i \in C} x_{ij} \leq |C| - 1 \tag{6}$$

is valid for P_{MKP}. In [2], Ferreira, Martin and Weismantel examine the question of when the multiple cover inequality is facet defining.

The strengthening idea for the multiple cover inequality follows the same idea as in the previous section. So, we refer to the same notation as in the previous section. Here, note that no more than $|J|(|S(c_1| + |S(c_2|)$ edges can be incident on the knapsacks in the multiple cover because otherwise at least one of the knapsacks would have more than $|S(c_1| + |S(c_2|)$ colors incident on it which in turn means that this knapsack has more than two colors incident on it. Thus, the inequality

$$\sum_{j \in J} \sum_{i \in C} x_{ij} \leq \min\{|C| - 1, |J|(|S(c_1)| + |S(c_2)|\} \tag{7}$$

is valid for MKCP.

4.3 (1,d)-configuration Inequalities

A set $N' \cup z$ with $N' \subseteq N$ and $z \in N \backslash N'$ is called a *(1,d)-configuration* with respect to some knapsack $k \in M$ if $\sum_{j \in N'} O_j \leq W_k$ and $K \cup \{z\}$ is a minimal cover with respect to knapsack k for all $K \subset N'$ with $|K| = d$. If $N' \cup z$ is a (1,d)-configuration with respect to some knapsack $k \in M$, the inequality

$$\sum_{i \in N'} x_{ik} + (|N'| - d + 1)x_{zk} \leq |N'|$$

is called a *(1,d)-configuration inequality* corresponding to $N' \cup z$. It was shown in [4] that this inequality defines a facet for $SK(N' \cup z, O, W_k)$.

Let c^* be the color corresponding to item z and let $c_1, c_2, ..., c_p$ be the colors represented in the item set N'. As before, we denote the sets of items corresponding to color c_i by $S(c_i)$. To strengthen the (1,d)-configuration inequality for MKCP, we consider two mutually exclusive cases:

1. No item in the set N' has the same color as item z. Then,

   - If $x_{zk} = 0$, we have

     $$\sum_{i \in N'} x_{ik} \leq \min\{d - 1, |S(c_1)| + |S(c_2|\}$$

     and hence

     $$\sum_{i \in N'} x_{ik} + (|N'| - d + 1)x_{zk} \leq \min\{d - 1, |S(c_1)| + |S(c_2|\} \tag{8}$$

   - If $x_{zk} = 1$, $\sum_{i \in N'} x_{ik} \leq \min\{d - 1, |S(c_1)|\}$ and hence

     $$\sum_{i \in N'} x_{ik} + (|N'| - d + 1)x_{zk} \leq \min\{d - 1, |S(c_1)|\} + |N'| - d + 1 \tag{9}$$

From (8) and (9), it then follows that the inequality

$$\sum_{i\in N'} x_{ik} + (|N'| - d + 1)x_{zk} \leq$$
$$\max\{\min\{d-1, |S(c_1)|\} + |N'| - d + 1, \min\{d-1, |S(c_1)| + |S(c_2|\}\}$$

is valid for MKCP.

2. Some item in the set N' has the same color as item z.

 - In this case, considering similar cases (i.e. $x_{zk} = 0$ and $x_{zk} = 1$) as above, we obtain that the inequality

$$\sum_{i\in N'} x_{ik} + (|N'| - d + 1)x_{zk} \leq$$
$$\max\{\min\{d-1, |S(c_1)| + |S(c_2)|\}, \min\{d-1, |S(c_1)| + |S(c^*)|\} + |N'| - d + 1\}$$

 is valid for MKCP.

4.4 Weight Inequalities

Recently, Weismantal [8] introduced the class of *weight inequalities* for the 0/1 knapsack problem. It is shown in [8] that these inequalities are needed to describe the knapsack polytope when the weight of the items lie in certain intervals. Here, we first define the weight inequalities and then provide a strengthening.

Let $T \subseteq N$ with $O(T) < W_k$ be given and let $r = W_k - O(T)$. Let I_j denote the set of items with weight j. Assuming $\bigcup_{j\geq r+1} I_j \backslash T \neq \phi$, the inequality

$$\sum_{i\in T} O_i x_i + \sum_{j\geq r+1} \sum_{i\in I_j} (j-r)x_i \leq O(T) \tag{10}$$

is called the *weight inequality with respect to* T [8].

Suppose the set T is composed of $p \leq N$ distinct colors $c_1, c_2, ..., c_p$. Let $S(c_1), S(c_2), ..., S(c_p)$ be the sets of items, respectively, corresponding to the colors $c_1, c_2, ..., c_p$. Without loss of generality, assume $|S(c_1)| \geq |S(c_2)| \geq ... \geq |S(c_p)|$. Then, as mentioned earlier in Section 4.1, no more that $|S(c_1)| + |S(c_2)|$ edges can be incident on knapsack k. Also, for a subset of items $Q \subseteq N$, let $O(Q)$ denote the weight of items in Q. Consider the disjunction

$$\sum_{j \geq r+1} \sum_{i \in I_j} x_i \leq 0 \bigvee \sum_{j \geq r+1} \sum_{i \in I_j} x_i \geq 1$$

In the first case,

$$\sum_{i \in T} O_i x_i + \sum_{j \geq r+1} \sum_{i \in I_j} (j-r) x_i \leq \sum_{i \in T} O_i x_i \leq O(S(c_1)) + O(S(c_2))$$

In the second case,

$$\begin{aligned} \sum_{i \in T} O_i x_i + \sum_{j \geq r+1} \sum_{i \in I_j} (j-r) x_i &= \sum_{i \in T} O_i x_i + \sum_{j \geq r+1} \sum_{i \in I_j} j x_i - r \sum_{j \geq r+1} \sum_{i \in I_j} x_i \\ &\leq W_k - (O(T) - (O(S(c_1)) + O(S(c_2)))) - r \\ &= O(S(c_1)) + O(S(c_2)) \end{aligned}$$

Thus, the inequality

$$\sum_{i \in T} O_i x_i + \sum_{j \geq r+1} \sum_{i \in I_j} (j-r) x_i \leq O(S(c_1)) + O(S(c_2)) \tag{11}$$

is valid for MKCP. Note that (11) is a strengthening of (10) whenever the set T consists of more than two colors.

5 Characteristics of basic fractional solutions to the linear programming relaxation LP_{MKCP}

In this section, we examine the structure of basic fractional solutions to LP_{MKCP}. The main result of the analysis is that a basic fractional solution can be eliminated by a minimal cover inequality associated with an individual knapsack constraint.

Let $\bar{x}$ be the optimum solution of the linear programming relaxation of (P). Define $R_G(\bar{V}, \bar{E})$, $\bar{V} \subseteq V$, $\bar{E} \subseteq E$, as the induced subgraph of $G(V, E)$ obtained by considering edges $(i, j) \in E$ such that $0 < \bar{x}_{ij} < 1$. In the following, we refer to $R_G(\bar{V}, \bar{E})$ as the *residual subgraph* of G. We consider two cases: (a) R_G does not contain any cycle and (b) R_G contains a cycle.

Case A: The residual graph does not contain any cycles.

Then, R_G is a forest and its connected components are trees. First, we assume that each connected component contains at least two knapsacks. We carry out the analysis in three steps: First, Lemma 5.1 shows the existence of a knapsack such that either its knapsack constraint is tight or one of its color constraints is tight. Then, Theorem 5.2 shows that if the knapsack is tight, then there exits a minimal cover inequality which cuts off $\bar{x}$. Theorem 5.3 shows that we will always find a knapsack such that its knapsack constraint is tight.

Lemma 5.1 *In any connected component (which is a tree), there exists at least one knapsack such that either its knapsack constraint or one of its color constraints is tight.*

Proof: Suppose not. That is, every knapsack has its knapsack constraint as well as all its color constraints slack. Consider a path P containing any two knapsacks K_i and K_j. The path P has the form K_i-I_{i_1}-K_j (see Figure 5). Keeping all other variables fixed, a perturbation of the fractional variables corresponding to P, $\bar{x}_{i_1,i} \pm \epsilon$ and $\bar{x}_{i_1,j} \pm \epsilon$ is feasible for LP_{MKCP} for some $0 < \epsilon < 1$. Since the knapsack constraint and the color constraints are slack for K_i and K_j, there exists a feasible perturbation $\epsilon^i > 0$, for variable $\bar{x}_{i_1,i}$ such that the knapsack constraint and the color constraints remain valid for K_i if $\bar{x}_{i_1,i}$ is perturbed to $\bar{x}_{i_1,i} \pm \epsilon^i$. Similarly, a feasible perturbation $\epsilon^j > 0$ exists for variable $\bar{x}_{i_1,j}$. Also, note that the proposed perturbations $\bar{x}_{i_1,i} \pm \epsilon$ and $\bar{x}_{i_1,j} \pm \epsilon$ also preserve the assignment constraint at item

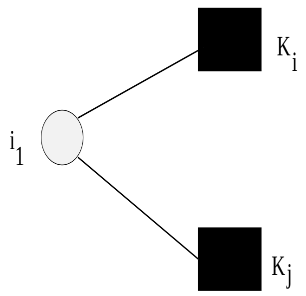

Figure 5: A path containing two knapsacks, Theorem 5.1

I_{i_1} so long as $\epsilon \leq \min\{x_{i_1,i}, 1 - x_{i_1,i}, x_{i_1,j}, 1 - x_{i_1,j}\}$. Hence, it is easy to see that choosing $\epsilon = \min\{x_{i_1,i}, 1 - x_{i_1,i}, x_{i_1,j}, 1 - x_{i_1,j}, \epsilon^i, \epsilon^j\}$ is sufficient. Note that $\epsilon > 0$.

Note that we have constructed two feasible solutions to LP_{MKCP}. However, the optimum solution $\bar{x}$ is a convex combination of these two feasible solutions which is a contradiction to the fact that $\bar{x}$ is a basic feasible solution. The result follows. $\square$

Theorem 5.2 *For a knapsack such that its knapsack constraint is tight, there exists a minimal cover inequality which cuts off the fractional solution.*

Proof: Let K_i be a knapsack whose knapsack constraint is tight. Let $I_{i_1}, I_{i_2}, ..., I_{i_p}$ be the items incident on K_i in R_G. Note that the $0 < x_{i_k,i} < 1$ for $k = 1, ..., p$. WLOG, assume that $O_{i_1} \geq O_{i_2} \geq ... \geq O_{i_p}$. Also, let $I_{j_1}, I_{j_2}, ..., I_{j_q}$ be the items, not in R_G, assigned to K_i. Note that $x_{j_k,i} = 1$ $k = 1, ..., q$. Let W_i be the capacity of knapsack K_i. Then,

$$\sum_{k=1}^{q} O_{j_k} + \sum_{k=1}^{p} O_{i_k} x_{i_k,i} = W_i$$

It is easy to see that $\cup_{k=1}^{q} I_{j_k} \bigcup \cup_{k=1}^{p} I_{i_k}$ is a cover for knapsack K_i. We construct a minimal cover, C, as follows: (a) Include items $I_{j_1}, I_{j_2}, ..., I_{j_q}$ in C and (b) Continue including items, starting from I_{i_1} till the knapsack capacity is exceeded. It is easy to see that the cover obtained is a minimal cover. So, suppose $\cup_{k=1}^{q} I_{j_k} \bigcup \cup_{k=1}^{l} I_{i_k}$ is the minimal cover where $l \leq p$. Then,

$$\sum_{k=1}^{q} O_{j_k} + \sum_{k=1}^{l} O_{i_k} > W_i \tag{12}$$

We claim that $\sum_{k=1}^{q} \bar{x}_{j_k,i} + \sum_{k=1}^{l} \bar{x}_{i_k,i} > q+l-1$. Because, if not then $\sum_{k=1}^{q} \bar{x}_{j_k,i} + \sum_{k=1}^{l} \bar{x}_{i_k,i} \leq q+l-1$ implies that $\sum_{k=1}^{q} O_{j_k} + \sum_{k=1}^{l} O_{i_k} \leq W_i$ which is a contradiction to (12).

Thus, the minimal cover inequality corresponding to the cover C is violated at knapsack K_i. This inequality cuts off the fractional solution $\bar{x}$. □

Theorem 5.3 *There always exists a knapsack such that its knapsack constraint is tight,*

Proof: By Lemma 5.1, there exists a knapsack such that either its knapsack constraint is tight or one of its color constraints is tight. Suppose the knapsack constraint is slack for all the knapsacks. Now consider a knapsack K_i one of whose color constraints is tight. We consider two cases:

1. Knapsack K_i has an edge e in R_G such that $\bar{x}_e$ appears in a tight color constraint for K_i.

   But then, because of the nature of the color constraint, there exists another edge e^* incident on K_i in R_G such that $\bar{x}_{e^*}$ appears in the same tight color constraint. Thus, we have two edges $e = (i_1, i)$ and $e^* = (j_1, i)$ in R_G. If there are any other knapsacks (other than K_i) incident to item i_1, we choose one of them. Similarly, if there are any other knapsacks (other than K_i) incident to item j_1, we choose one of them. Thus, starting from K_i, we continue building a path in two directions (see Figure 6). Note that we will necessarily end up with a path (for otherwise, R_G would contain a cycle). An end point of the path, P, thus obtained can be an item or a knapsack (see Figure 6 for possible configurations). Note that all the knapsacks in P have the knapsack constraint slack.

- If both the endpoints of P are knapsacks, then P has the form

$$P = K_{i_p^*} - i_p - K_{i_{p-1}^*} \ldots i_2 - K_{i_1^*} - i_1 - K_i - j_1 - K_{j_1^*} - j_2 - \ \ldots \ - j_q - K_{j_q^*}$$

  Then, note that the variables $\bar{x}_{i_p,i_p^*}$ and $\bar{x}_{i_q,i_q^*}$ corresponding to the two endpoints of P are fractional and hence cannot be in any tight color constraint for knapsacks $K_{i_p^*}$ and $K_{j_q^*}$ respectively. Hence, there exists $\epsilon > 0$ such that the two perturbations

$$(\bar{x}_{i_p,i_p^*} \pm \epsilon, \bar{x}_{i_p,i_{p-1}^*} \mp \epsilon, \ldots, \bar{x}_{i_1,i} \pm \epsilon, \bar{x}_{j_1,i} \mp \epsilon, \ldots, \bar{x}_{j_q,j_q^*})$$

  generate two feasible solutions (all other variables remain at their original value) for LP_{MKCP}. But, since $\bar{x}$ is a convex combination of these two solutions, we have a contradiction to the fact that $\bar{x}$ is a basic feasible solution.

- If one or both the end points of P are items, then note that the assignment constraint cannot be tight at the item(s) at the endpoint and hence a similar argument provides a contradiction to the fact that $\bar{x}$ is a basic feasible solution.

2. Knapsack K_i does not have any edge in R_G which appears in a tight color constraint. Then, we consider a path P from knapsack K_i to a leaf node of the tree. That is, one end point of P is knapsack K_i while the other end point of P is a leaf node of the tree and can either be an item or a knapsack. In either case, a similar argument shows a contradiction to the fact that $\bar{x}$ is a basic feasible solution.

Suppose every tree in the forest contains only one knapsack. For a tree, if the only knapsack K it contains has degree one in R_G, then it is easy to see that $\bar{x}$ is not an optimum solution. If the degree of K is greater than one, a similar argument as above provides a contradiction to the fact that $\bar{x}$ is a basic feasible solution.

Case B: The residual graph contains a cycle.

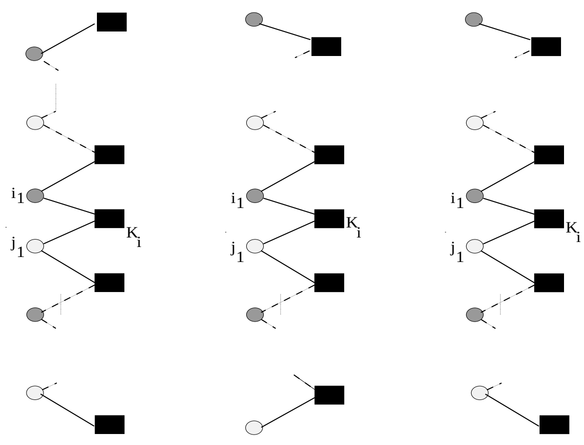

Figure 6: Possible configurations for case 1, Theorem 5.3

Consider a cycle C in the residual graph. We consider two possibilities:

1. C contains a knapsack whose knapsack constraint is tight: Here, Theorem 5.2 is again applicable and hence we can find a minimal cover inequality which cuts off the fractional solution $\bar{x}$.

2. C does not contain any knapsack with a tight knapsack constraint: Since each node of C has exactly two edges of C incident on it. For every node, of the two *fractional* variables (say having values f_1 and f_2) corresponding to the two edges incident on it, we perturb the value of one variable (say f_1) as $f_1 \pm \epsilon$ and the value of the other variable as $f_1 \mp \epsilon$. Since, C is of even length, it is easy to see after such a perturbation for the edges of the cycle, every node will have one edge whose value has been decreased by ϵ and one edge whose value has been increased by ϵ. Notice that, this perturbation does not violate the assignment constraints for the items and the color constraints for the knapsack. However, the knapsack constraints might be violated. But note that the knapsack constraints are *slack*. So, there exists some $\epsilon > 0$ such that the perturbations above are feasible for MKCP (values of the variables not in C are not changed).

We have thus demonstrated two feasible solutions for MKCP such that $\bar{x}$ is a convex combination of these two solutions. This contradicts the fact that $\bar{x}$ is a basic feasible solution.

6 Extension to more than two colors

For the problem considered in this paper, the number of distinct colors which could be assigned to a knapsack is bounded above to two. For the case when the number of distinct colors allowed is more than two, some of the results in Sections 3 and 4 can be extended. For example, Theorems 3.5 and 3.9, which characterize when a color constraint is redundant, can be easily extended. Also, the results in Section 4 regarding the strengthening of valid inequalities can be easily extended. However, the proofs of Theorems 3.6, 3.7, 3.8, 3.10 and 3.11, which describe necessary conditions and sufficient conditions for a color constraint to be facet defining, make specific use of the fact that the number of distinct colors is bounded above by 2 and hence it is not trivial to extend them.

7 Summary and Conclusions

In this paper, we studied the multiple knapsack problem with color constraints (MKCP). Motivated by our experience in the steel industry, the MKCP generalizes the multiple knapsack problem by adding a "color" attribute to each order and including the associated color constraints which restrict the number of allowable colors in any knapsack to be no more than two. Such constraints capture a large class of processing constraints in the steel industry.

We presented three main theoretical results in this paper. The first one provides the necessary and sufficient conditions under which the newly introduced color constraints are redundant. The second set of results provide some sufficient conditions and some necessary conditions for a color constraint to be a facet. The third result shows that a basic fractional solution obtained by solving the LP relaxation can be cut off by a minimal cover inequality. We also provide results which consider the color constraints to strengthen four classes of valid inequalities which have been well studied for the MKP.

Two directions for future research seem promising: (i) Identifying a set of necessary and sufficient conditions under which a facet for MKP is also a facet for MKCP, and (ii) incorporating the results of this paper in a computational framework such as branch-and-cut.

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